Ambiguities in the retrieval of' rain rates from radar returns at attenuating wavelengths

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Abstract

It is well-known that there are significant deterministic ambiguities inherent in trying to determine the particular rain rate profile which produced some given sequence of air- or space-borne radar echo powers at a single attenuating frequency. We quantify these ambiguities mathematically, and examine their effect on various proposed rain-rate profile retrieval algorithms. When the given data consist of a single radiometer measurement together with a single-look-angle single-frequency set of range-compressed echo powers, we show that several substantially different rain profiles can realistically be considered solutions. On the other hand, if the data consist of a single-look-angle two-frequency set of echo powers, the inversion problem generically has a unique solution. We note that traditional "back-of-the-envelope" arguments can be quite misleading in assessing the extent of the ambiguity, even in the simplest cases.

1 Introduction

The problem of estimating the vertical rainfall profile from measurements obtained using an active narrow-band air- or space-borne radar has already been studied extensively (see for example Atlas and Ulbrich 1977, Fujita 1983, Hitschfeld and Bordan 1954, Marzoug and Amayenc 1991, Meneghini 1978, Meneghini and Nakamura 1990, Weinman et al 1990, among many others). In order to approach the problem mathematically, one needs to model the dependence of the received power on the rain rate itself, Traditionally, one then proceeds to count the number of equations and the number of unknown variables, assuming the variables in the model have been discretized: if the number of equations exceeds the number of variables, conventional wisdom dictates that an unambigous solution must exist. On the other hand, if there are more unknowns than equations, one might look for additional constraints that might force a unique solution. There are two important drawbacks to such an approach. The main one is that there is no guarantee that the equations one gets will be independent (indeed, as we shall see, in some simple realistic cases, they are not). A more basic shortcoming is that the discretization is an artifact that is necessary only for the implementation of an actual estimation algorithm. One would be closer to reality if one kept in mind that the received power is a function of continous time, and the rain rate a function of continuous range. The most notable result using a continuous model is the one obtained by Hitschfeld and Bordan (1954). It suggests that uncertainties in the radar calibration can lead to sizable errors. Meneghini (1976') studied the effect of small uncertainties in some of the parameters relating the rain rate to the measured radar reflectivies, and showed that serious errors could result when one uses various inversion algorithms. In this paper, we investigate the ambiguities in the rain rate profile retrieval problem that are due to the varying nature of the rainfall itself. These inherent errors are *in addition* to the errors due to the eventual instability of the inversion algorithm used, and to the random noise-induced fluctuations. Indeed, as we show in this paper, these ambiguities *do not* disappear if one uses a numerically stable inversion method, such as the surface-reference technique proposed by Meneghini and Nakamura (1990), or the one proposed by Marzoug and Amayenc (1991).

A more subtle complication is the fact that the ambiguity issue is a scientific one rather than a purely mathematical one: indeed, one need not be concerned about the uniqueness of an eventual solution, but rather about the difference between multiple solutions, if any. For example, while the single-frequency case turns out to be unacceptably ambiguous, additional measurements with a different look angle turn the problem into one which admits non-unique solutions that are *not* significantly different. We will use simple continuous models similar to the one used by Hitschfeld and Bordan (1954) to examine these issues, starting with the single-frequency case. In the examples, we shall be most interested in the case where the single frequency is $\simeq$ 13.8 GHz, the frequency of the Precipitation Radar of the Tropical

Rainfall Measuring Mission (TRMM - see Theon and Fugono, 1988).

2 Single frequency -- Hitschfeld-Bordan approach

Let us begin by reviewing the classical result of Hitschfeld and Bordan's. We use the simple model that the calibrated effective reflectivity $p(r)$ measured by a nadir-looking monostatic narrow- band radar, from range r, is proportional to the reflectivity coefficient $Z(r)$ of the rain at range r, and to the accumulated attenuation from range 0 to range r, assuming that $r = 0$ is the range at the top of the rain cell. Calling $k(r)$ the attenuation coefficient at range r, and in the absence of noise, this is equivalent to assuming that the calibrated reflectivity is exactly given by

$$p(r) = Z(r)10^{-0.2\int_0^r k(t)dt}. \tag{1}$$

Following Hitschfeld and Bordan, let us assume that we are given p and need to determine Z and k. Empirically, it seems reasonable to assume that Z and k are related to the rain rate $R(r)$ by equations of the form $Z = aR^b$ and $k = \alpha R^\beta$ where the parameters a, b, α and β are to be determined. So we substitute this expression for Z and k in (1), and proceed to solve for R. As in (Hitschfeld and Bordan 1954), the solution is

$$R(r) = \left(\frac{(p(r)/a)^{\beta/b}}{1 - \frac{0.2\alpha\beta\log(10)}{b}\int_0^r (p(t)/a)^{\beta/b}dt}\right)^{1/\beta} \tag{2}$$

Thus, if a, b, α and β were known, (2) would determine R uniquely. Indeed, recent studies show that if three of these four parameters are assumed known a priori, and if one extra measurement such as a radiometer reading is used to determine the remaining parameter, then one can calculate the underlying rain rate R exactly.

Since it is highly unlikely that a, b, α and β can be known exactly, there remains to quantify the effect on R of an error in a, b, α or β. Indeed, *any* value of (a, b, α, β) will give a rain rate profile R whose resulting returns will amount to the measured effective reflectivity. Just how different can two R's obtained using two different α's get? To answer sue.]) questions, we start with the empirical ly-verified simplifying assumption that the rain reflectivity and attenuation coefficient are indeed related to the rain rate by power laws. Thus, the calibrated effective reflectivity $p(r)$ received from range r is related to the rain rate $R(t)$ at ranges $t \leq r$ by

$$p(r) = aR(r)^b 10^{-0.2\alpha\int_0^r R(t)^\beta dt}, \tag{3}$$

where a, b, α and β are parameters to be determined. Assuming that these parameters remain constant throughout the rain column (a convenient simplifying assumption which we shall do away with in section 4), we are interested in finding out if it is possible that two different sets of rainfall parameters $\{R_0(r), a_0, b_0, \alpha_0, \beta\}$ and $\{R_1(r), a_1, b_1, \alpha_1, \beta_1\}$ give rise to the same effective reflectivity $p(r)$, and, in that case, how different R_0 and R_1 can be.

As a first step, let us consider a simple version of this problem: suppose that a, b, and β are fixed and assumed known, that we are given $R_0(r)$ and α_0, and let us try to determine under what conditions a different $R_1(r)$ and α_1 could still give rise to the same effective reflectivity profile, i.e. would satisfy

$$aR_1(r)^b 10^{-0.2\alpha_1 \int_0^r R_1(t)^\beta dt} = aR_0(r)^b 10^{-0.2\alpha_0 \int_0^r R_0(t)^\beta dt}, \tag{4}$$

at all ranges r. Writing F for the ratio R_1/R_0, interchanging terms on either side of the equality, then taking the logarithm of the two sides, one can transform (4) into

$$\log F(r) = \frac{-0.2\alpha_0 \log(10)}{b} \int_0^r \left(1 - \frac{\alpha_1}{\alpha_0} F(t)^\beta\right) R_0(t)^\beta dt, \tag{5}$$

which, when differentiated with respect to r, becomes

$$\frac{F'(r)}{F(r)} = \frac{-0.2\alpha_0 \log(10)}{b} \left(1 - \frac{\alpha_1}{\alpha_0} F(r)^\beta\right) R_0(r)^\beta, \tag{6}$$

in which F' denotes the derivative of F with respect to range. Eq. (6) can be rewritten as

$$\frac{F'}{F\left(1 - \frac{\alpha_1}{\alpha_0} F^\beta\right)} = \frac{-0.2\alpha_0 \log(10)}{b} R_0^\beta \tag{7}$$

which, when integrated with respect to range, gives

$$-\frac{1}{\beta} \log\left(\frac{F(r)^{-\beta} - \alpha_1/\alpha_0}{1 - \alpha_1/\alpha_0}\right) = \frac{-0.2\alpha_0 \log(10)}{b} \int_0^r R_0(t)^\beta dt. \tag{8}$$

We can now solve for F, and find that for *any* value of α_1, there is a new rain rate profile $R_1(r)$, namely

$$R_1(r) = \frac{R_0(r)}{\left(\frac{\alpha_1}{\alpha_0} + \left(1 - \frac{\alpha_1}{\alpha_0}\right) 10^{\frac{0.2\beta\alpha_0}{b} \int_0^r R_0(t)^\beta dt}\right)^{1/\beta}} \tag{9}$$

which can be responsible for the same effective reflectivity power as R_0.

A convenient rough measure of how different such an R_1 is from the original R_0 is the value of the non-negative quantity $Q(r) = \left(\frac{R_1(r)}{R_0(r)} - 1\right)^2$, the square of the relative difference between R_1 and R_0 (the closer Q is to zero, the smaller the difference). It is encouraging that, in this simple case where only the parameter α is not assumed known, $Q(0) = 0$. We seem to be off to a good start: the difference is initially tiny. To get an idea of its evolution with increasing range, one can compute its derivative $\frac{dQ}{dr}$. Under the additional simplifying assumption that $\beta = 1$ (an empirically reasonable value), one finds that

$$\frac{dQ}{dr} = \frac{2g'(r)(1-x)^2}{[(x+(1-x)g(r))^2]} \left[\frac{g(r)-1}{x+(1-x)g(r)}\right], \tag{10}$$

where $x = \frac{\alpha_1}{\alpha_0}$, $g(r) = 10^{\frac{0.2\alpha_0\beta}{b}\int_0^r R_0(t)dt}$, $g'(r) = \frac{0.2\alpha_0\beta\log(10)}{b} g(r)$. The terms within the first brackets are all positive. The numerator in the second bracket is positive as soon as $r > 0$. The denominator is positive at $r = 0$ and it is continuous, hence, as long as it doesn't vanish (and we have been implicitly assuming it remains non-zero over the realistic range), the denominator remains positive also. Thus Q is monotonically increasing as a function of range. This means that while the difference between R_1 and R_0 always starts out small, it must increase steadily with range. In fact, given any value $\epsilon > 0$, one can determine the range r_ϵ at which $Q(r_\epsilon)$ will reach the value ϵ^2: indeed, one finds that r_ϵ would have to satisfy the equation

$$r_\epsilon = \begin{cases} \frac{b}{0.2\alpha_1 R_0^{avg}\log(10)} \log\left(\frac{1-x+x\epsilon}{(1-\epsilon)(1-x)}\right) & \text{if } \alpha_1 > \alpha_0 \\ \frac{b}{0.2\alpha_1 R_0^{avg}\log(10)} \log\left(\frac{1-x+\epsilon}{(1+\epsilon)(1-x)}\right) & \text{if } \alpha_1 < \alpha_0 \end{cases} \tag{11}$$

where $R_0^{avg} = \frac{1}{r_\epsilon}\int_0^{r_\epsilon} R_0(t)dt$ is the current average value of the rain rate. One can see from (11) that Q can exceed the value 1/4 very quickly, allowing us to produce a rain profile that quickly decreases to one-half the value of the original rain profile while still producing the same effective reflectivity. Figure 1 shows just such an example: we chose $R_0 = 40$ mm/hr, $b = 1.5$, $\beta = 1$, $\alpha_0 = 0.036$, then computed R_1 according to (9) with $\alpha_1 = 0.018$. These values are well within the documented bounds for the parameters in this model (see, for example, Atlas and Ulbrich 1974, Battan 1973, Ulbrich 1983 note that our k is Ulbrich's A) for radar frequencies near 13.8 GHz, the frequency of the TRMM radar. As is evident from the graph, R_1 reaches 20 mm/hr at range $r_{1/2} \approx 3.3$ km. Figure 2 shows the case where R_0 is still assumed constant, equal to 20 mm/hr this time, and $\alpha_0 = 0.018$, $\alpha_1 = 0.036$: R_1 can again be computed from (9), and in this case the denominator shrinks quickly to zero after a few kilometers. Figure 3 shows what happens in a case where the original profile R_0 is not constant. Indeed, the profile used as R_0 in this example is essentially the fifth sample profile studied by Fujita (1983), and is represented by the dashed curve. Taking $\alpha_0 = 0.018$, $\alpha_1 = 0.036$ produces a new rain rate profile R_1 which quickly diverges by over 100% from the original. Figure 4 shows what happens with the same R_0, but with

$\alpha_1 = 0.018$, $\alpha_0 = 0.036$. In all four cases, the relative error dots indeed grow steadily with range, to unacceptable levels.

Let us proceed to the general case: given rain data $\{R_0(r), a_0, b_0, \alpha_0, \beta_0\}$, we are looking for $\{R_1(r), a_1, b_1, \alpha_1, \beta_1\}$ such that

$$a_1 R_1(r)^{b_1} 10^{-0.2\alpha_1 \int_0^r R_1(t)^{\beta_1} dt} = a_0 R_0(r)^{b_0} 10^{-0.2\alpha_0 \int_0^r R_0(t)^{\beta_0} dt}. \tag{12}$$

We transform this equation by

1) dividing both sides by a_1,

2) raising both sides to the power β_1/b_1, then

3) multiplying both sides by $-0.2\log(10)\alpha_1\beta_1/b_1$.

The equation then becomes

$$\frac{d}{dr}\left(10^{-0.2\alpha_1 \int_0^r R_1(t)^{\beta_1} dt}\right) = \frac{-0.2\log(10)\alpha_1\beta_1}{b_1}\left(\frac{a_0}{a_1}\right)^{\beta_1/b_1} R_0(r)^{b_0\beta_1/b_1} 10^{-0.2\frac{\alpha_0\beta_1}{b_1}\int_0^r R_0(t)^{\beta_0} dt}. \tag{13}$$

Integrating both sides gives

$$10^{-0.2\alpha_1 \int_0^r R_1(t)^{\beta_1} dt} = 1 - \frac{0.2\log(10)\alpha_1\beta_1}{b_1}\left(\frac{a_0}{a_1}\right)^{\beta_1/b_1} \int_0^r R_0(r')^{b_0\beta_1/b_1} 10^{-0.2\frac{\alpha_0\beta_1}{b_1}\int_0^{r'} R_0(t)^{\beta_0} dt}\, dr'. \tag{14}$$

Taking the decimal logarithm of both sides, differentiating with respect to r, then simplifying the result, we finally obtain

$$R_1(r) = \frac{R_0(r)^{b_0/b_1} 10^{-0.2\frac{\alpha_0}{b_1}\int_0^r R_0(t)^{\beta_0} dt}}{\left(\left(\frac{a_1}{a_0}\right)^{\beta_1/b_1} - \frac{0.2\log(10)\alpha_1\beta_1}{b_1}\int_0^r R_0(r')^{b_0\beta_1/b_1} 10^{-0.2\frac{\alpha_0\beta_1}{b_1}\int_0^{r'} R_0(t)^{\beta_0} dt}\, dr'\right)^{1/\beta_1}} \tag{15}$$

This is the equation we have been looking for. Before using it, let us check that it reduces to (9) in the special case where only α is allowed to vary. Indeed, in that case, (15) can be rewritten as

$$R_1(r) = \frac{R_0(r) 10^{-0.2\frac{\alpha_0}{b}\int_0^r R_0(t)^{\beta} dt}}{\left(1 + \frac{\alpha_1}{\alpha_0}\int_0^r \frac{-0.2\log(10)\alpha_0\beta}{b} R_0(r')^{\beta} 10^{-0.2\frac{\alpha_0\beta}{b}\int_0^{r'} R_0(t)^{\beta} dt}\, dr'\right)^{1/\beta}} \tag{16}$$

$$= R_0(r)\left(\frac{10^{-0.2\frac{\alpha_0\beta}{b}\int_0^r R_0(t)^{\beta} dt}}{1 + \frac{\alpha_1}{\alpha_0}\left(10^{-0.2\frac{\alpha_0\beta}{b}\int_0^r R_0(t)^{\beta} dt} - 1\right)}\right)^{1/\beta}, \tag{17}$$

which is indeed equivalent to (9).

It turns out that the new rain rates R_1 obtained using (15) can be at least as ill-behaved (from our perspective) as those encountered earlier in the special case. Indeed, figure 5 shows the case where R_0 is as in figure 4, but with parameters selected as follows: $a_0/a_1 = 0.73, b_0 = 1.24, b_1 = 1.8, \alpha_0 = 0.018, \alpha_1 = 0.036, \beta_0 = 0.98, \beta_1 = 1.18$. The difference between R_0 and R_1 is substantial. Another example is shown in figure 6, where the parameters are: $a_0/a_1 = 1.03, b_0 = 1.34, b_1 =$ *1.64,* C% $= 0.018, \alpha_1 =$ *0.036,/3₀= 0.9 S,/?₁*= 1.18. Once again, the difference is very substantial. This example is perhaps more remarkable Ijc.cause both profiles actually share the same average ($\simeq$ 50 mm/hr). Thus, even if we had a reliable estimate of the average amount of rain over the whole column under study in that case, we would still not be able to decide between R_0 and R_1 (and the continuum of possibilities between and around them).

3 Single frequency -- Surface-Reference approach

Before going on to less simplified rain reflectivity models, let us examine the ambiguities when, instead of using the directly measured effective reflectivities as our starting point, we use surface-referenced data as proposed by Marzoug and Amayenc (1991) instead, i.e. if we divide $p(r)$ at every range r by $p(r_s)$, where r_s denotes the range to the surface, Writing o for the surface backscattering coefficient, a derivation identical to the one we used above to obtain (15) shows that two sets of rain data $\{R_0(r), a_0, b_0, \alpha_0,$ / 30 , 0 , $\}$ and $\{R_1(r), a_1, b_1, \alpha_1, \beta_1, \sigma_1\}$ give rise to the same surface-referenced effective reflectivities if

$$R_1(r) = \frac{R_0(r)^{b_0/b_1} 10^{0.2\frac{\alpha_0}{b_1}\int_r^{r_s} R_0(t)^{\beta_0} dt}}{\left(\left(\frac{\sigma_0 a_1}{\sigma_1 a_0}\right)^{\beta_1/b_1} + \frac{0.2\log(10)\alpha_1\beta_1}{b_1}\int_r^{r_s} R_0(r')^{b_0\beta_1/b_1} 10^{0.2\frac{\alpha_0\beta_1}{b_1}\int_{r'}^{r_s} R_0(t)^{\beta_0} dt} dr'\right)^{1/\beta_1}}. \qquad (18)$$

By varying the various parameters in (18), one can again easily produce different rain profiles that give the same surface-l(:fclcl)cccl echo profile. This highlights the fact that these ambiguities are inherent in the single-frequency problem itself, and are not artifacts of the inversion method used. Moreover, as in the Hitschfeld-Bordan approach, the error due to these ambiguities manifestly contributes exponentially with range.

We illustrate this effect by considering a constant rain rate profile $R_0 = 20$ mm/hr, and using (18) to compute two profiles that would have produced the same surface-referenced effective reflectivities, in the two cases where

1) $\alpha_0 = 0.02$, $\alpha_1 = 0.03$, $\beta_0 = 0.98$, $\beta_1 = 1.08$, $b_0 = 1.4$, $b_1 = 1.6$, and $\frac{\sigma_0}{\sigma_1} = 1.5\frac{a_0}{a_1}$

2) $\alpha_0 = 0.03$, $\alpha_1 = 0.02$, $\beta_0 = 1.08$, $\beta_1 = 0.98$, $b_0 = 1.6$, $b_1 = 1.4$, and $\frac{\sigma_0}{\sigma_1} = 0.75$.

Figure 7 shows the two corresponding profiles. Assuming that we have perfect knowledge of the (possibly raill-modified) surface backscattering coefficient, i.e. assuming that $\sigma_0 = \sigma_1$, these two cases show how a relatively small change in the parameter a can kead to significantly different associated rain rate profiles. Case 1 corresponds to the situation where $a_1 = 1.5a_0$, which happens for example when $a_0 = 250$ and $a_1 = 375$ (the units are such that Z is expressed in $mm^6 \cdot m^{-3}$). The resulting rain rate R_1 is almost 100 % below R_0 = 20 mm/hr. Note that, in this case, the total path-integrated attenuation is almosst the same for the two profiles: 1.3 dB for the profile R_0, and 0.98 dB for R_1. This is significant because, in practice, when one has an estimate of the surface backscattering coefficient σ to use in the surface-reference method, one would have a (more or less ac.curate) estimate of the path-inegrated attenuation as well, and that estimate could be used to refine one's estimates (as is proposed in Meneghini and Nakamura, 1 990). This first example shows that even in this case, significantly ambiguous profiles can still exist, with no exploitable difference in the total path-integrated attenuation. As to case 2, it corresponds to the situation where $a_1 = 0.75a_0$, which happens for example when $a_0 = 400$ and $a_1 = 300$. These values for the coefficient a are well within the published ranges found by regression analysis (Battan 1973, Ulbrich 1983). The profile R_1 in this case is almost 100 % greater than R_0. Thus, even when the surface backscattering coefficient is known exactly, the surface-reference rain retrieval method still has large inherent ambiguities.

One can interpret these two examples in a different way, assuming that the parameter a is known exactly (i.e. that $a_0 = a_1$). Specifically, these two cases show how a relatively small change in the value of the surface backscattering coefficient σ can lead to significantly different derived rain rate profiles. Indeed, with $a_0 = a_1$, the first case shows that a 1.7 dB decrease in σ_0 (which corresponds to $\sigma_0/\sigma_1 = 1.5$), along with small changes in the remaining parameters, results in an underestimate of the true rain rate $R_0 = 20$ mm/hr that is almost 100 % short of the correct value. Similarly, the second case shows that increasing σ_0 by 1.2 dB (so that $\sigma_0/\sigma_1 = 0.75$) produces an overestimate of the true rain rate. Allowing for simultaneous uncertainties in the parameter a only aggravates the ambiguity.

4 Single frequency – Drop-Size-Distribution approach

Let us now remove the assumption] that the $Z-R$ and $k-R$ relations are constant throughout the rain column. Rather than allow these quantities to vary arbitrarily, we express Z, k and R directly in terms of the drop size distribution. Indeed, if $N(D)\,dD$ denotes the number of drops per cubic meter whose diameter is between D and $D + dD$ mm, then

$$Z = \int D^6 N(D)\, dD \;\; mm^6/m^3 \;\; \text{(in the Rayleigh approximation)}, \tag{19}$$

$$k = \int 4343\, \sigma_t(D)\, N(D)\, dD \;\; dB/km, \tag{20}$$

$$R = \int 6\pi\, 10^{-4}\, v(D)\, D^3\, N(D)\, dD \;\; mm/hr, \tag{21}$$

where $\sigma_t(D)$ is the total absorption-and-scattering cross-section for a drop of diameter D, and $v(D)$ is the drop fall velocity in m/sec. To make these expressions practically useful, we need to replace $N(D)$ by a physically reasonable analytical expression, then preform the required integrations. Following Ulbrich (1983), we shall assume that $N(D)$ is Γ-distributed, i.e. that

$$N(D) = N_T D^{m-1} e^{-D/(\overline{D}/m)}, \quad 0 < D < \infty \tag{22}$$

with N_T, m and $\overline{D}$ the parameters of the distribution. In this notation, $\overline{D}$ denotes the average drop diameter and m is the "curvature" parameter of the distribution. For $\sigma_t(D)$ and $v(D)$, Atlas and Ulbrich (1974 and 1977) have shown that power-law approximations are adequate provided the coefficients are appropriately chosen. Using the relations

$$4343\;\; \sigma_t(D) \simeq 1.85 \cdot 10^{-4}\, D^{4.273} \tag{23}$$

at 13.8 GHZ (in the Rayleigh approximation, see Atlas and Ulbrich 1974, Kozu 1991), and

$$v(D) \simeq 3.78 D^{0.67} \;\; m/sec \tag{24}$$

(see Atlas and Ulbrich 1977) our three quantities Z, k and R are given by

$$Z \simeq \Gamma(m+6)\left(\frac{\overline{D}}{m}\right)^{m+6} N_T, \tag{25}$$

$$k \simeq 1.85 \cdot 10^{-4}\, \Gamma(m+4.27)\left(\frac{\overline{D}}{m}\right)^{m+4.27} N_T, \tag{26}$$

$$R \simeq 7.1 \cdot 10^{-3}\, \Gamma(m+3.67)\left(\frac{\overline{D}}{m}\right)^{m+3.67} N_T. \tag{27}$$

Our $Z - R$ and $k - R$ relations can now be expressed as

$$Z = 140.8 \cdot \frac{\Gamma(6+m)}{\Gamma(3.67+m)}\left(\frac{\overline{D}}{m}\right)^{2.33} \cdot R, \tag{28}$$

$$k = 0.026\, \frac{\Gamma(4.27+m)}{\Gamma(3.67+m)}\left(\frac{\overline{D}}{m}\right)^{0.6} \cdot R, \tag{29}$$

and our problem is to find $R(r)$, $m(r)$ and $\overline{D}(r)$, given our measured (calibrated) radar reflectivities $p(r)$. It is important to point out that, although (28) and (29) seem to imply linear relations between k, Z and R, this is not the case. Indeed, we are not assuming that the variables R, m and $\overline{D}$ are independent (in fact data suggests that $\overline{D}$ and R are closely correlated), or that their covariances cannot change with range. once it is noticed that R, m and $\overline{D}$ may be closely realted to one another, and that these relations may change with range, it becomes clear that formulas (28) and (29) can indeed be approxiamted by power laws under appropriate additional assumptions.

Under these assumptions, and using this notation, the problem is: given rain data $\{R_0(r), m_0(r), \overline{D}_0(r)\}$, are there any different data $\{R_1(r), m_1(r), \overline{D}_1(r)\}$ satisfying

$$140.8 \cdot \frac{\Gamma(m_1(r)+6)}{\Gamma(m_1(r)+3.67)}\left(\frac{\overline{D}_1(r)}{m_1(r)}\right)^{2.33} \cdot R_1(r) \cdot 10^{-0.2\int_0^r 0.026 \cdot \frac{\Gamma(m_1(t)+4.27)}{\Gamma(m_1(t)+3.67)} \cdot \left(\frac{\overline{D}_1(t)}{m_1(t)}\right)^{0.6} \cdot R_1(t)\,dt} \tag{30}$$

$$= 140.8 \cdot \frac{\Gamma(m_0(r)+6)}{\Gamma(m_0(r)+3.67)}\left(\frac{\overline{D}_0(r)}{m_0(r)}\right)^{2.33} \cdot R_0(r) \cdot 10^{-0.2\int_0^r 0.026 \cdot \frac{\Gamma(m_0(t)+4.27)}{\Gamma(m_0(t)+3.67)} \cdot \left(\frac{\overline{D}_0(t)}{m_0(t)}\right)^{0.6} \cdot R_0(t)\,dt}$$

and, if so, how different can R_0 and R_1 be?

To answer this question, we shall use the same mathematical derivation that gave us (15), starting with (30) this time. Specifically, we

1) multiply both sides of (30) by $-0.0052\log(10)\frac{\Gamma(4.27+m_1)}{\Gamma(6+m_1)}\left(\frac{m_1}{\overline{D}_1}\right)^{1.73}$,

2) integrate from 0 to r,

3) isolate $10^{-0.2\int_0^r 0.026 \cdot (\Gamma(m_1(t)+4.27)/\Gamma(m_1(t)+3.67)) \cdot (\overline{D}_1(t)/m_1(t))^{0.6} \cdot R_1(t)\,dt}$ on the left-hand-side,

4) take the decimal logarithm of both sides,

5) differentiate with respect to r and simplify the result.

When applied consecutively, these steps will transform (30) into

$$R_1(r) = R_0(r)\frac{\left(\frac{m_1(r)\overline{D}_0(r)}{m_0(r)\overline{D}_1(r)}\right)^{0.6} F(r)\,10^{-0.2\int_0^r k_0(t)\,dt}}{F(r)\,10^{-0.2\int_0^r k_0(t)\,dt} + (1-F(0)) - \int_0^r F'(t)\,10^{-0.2\int_0^t k_0(\tau)\,d\tau}\,dt} \tag{31}$$

where $F(r) = \frac{\Gamma(m_0(r)+6)\Gamma(m_1(r)+4.27)}{\Gamma(m_1(r)+6)\Gamma(m_0(r)+4.27)}\left(\frac{m_1(r)\overline{D}_0(r)}{m_0(r)\overline{D}_1(r)}\right)^{1.73}$, $F'(r)$ denotes the derivative of F with respect to r, and $k_0(r)$ is the attenuation coefficient for the profile $R_0(r)$ as in (29). The most striking resemblence between this equation and its power-law counterparts (15) and (18) is the fact that the ambiguities contribute exponentially with range.

To illustrate the usefulness of (31), we shall determine the different ambiguous profiles that would produce the same effective reflectivity profile as a constant $R_0 = 20$ mm/hr, using the drop-size-distribution model. Before we can apply (31), we must first determine the realistic range of values that we need to consider for m and $\overline{D}$ for rain rates of about 20 mm/hr. To do that, we use the documented empirical analyses of the relation between observed values of Z, k and R, when R is near 20 mm/hr. Since these regression analyses produce power-law $Z - R$ and $k - R$ relations, we shall try to rewrite (28) and (29) as $Z = aR^b$ and $k = \alpha R^\beta$ respectively, with $b = 1.5$, $\beta = 1$, and $R = 20$ mm/hr, and try to determine the values of m and $\overline{D}$ that produce values of a and α within the range observed by the empirical analyses (see Battan 1973, Kozu 1991, Kozu and Nakamura 1991, Ulbrich 1983). The values of m and $\overline{D}$ that we considered, along with the corresponding values of a and α for $R = 20$ mm/hr, are shown in table 1. Based on these values, let us examine the

m	$\overline{D}$	a	α
1/2	1/8	48	0.026
1/2	1/4	241	0.0393
1/2	1/2	1215	0.059
1	1/4	60	0.0278
1	1/2	305	0.042
1	3/4	784	0.054
2	1/2	91	0.031
2	1	457	0.0477
2	2	2298	0.072
3	1/2	50	0.027
3	1	250	0.0412
3	2	1260	0.0625
4	1	173	0.038
4	3/2	445	0.048
4	2	870	0.057

Table 1: DSD parameters ↔ power-law correspondence at 20 mm/hr

following four cases:

- case 1: $m_0 = 4$, $\overline{D}_0 = 1.5$, $m_1 = 0.5$, $\overline{D}_1 = 0.25$.
- Case 2: $m_0 = 0.5$, $\overline{D}_0 = 0.25$, $m_1 = 4$, $\overline{D}_1 = 1.5$.
- case 3: $m_0 = 4$, $\overline{D}_0 = 1.5$, $m_1 = 2$, $\overline{D}_1 = 1$.
- case 4: $m_0 = 2$, $\overline{D}_0 = 1$, $m_1 = 4$, $\overline{D}_1 = 1.5$.

Since m and $\overline{D}$ are not likely to remain constant over long segments of the rain column, we restricted our attention to ranges between $r = 0$ and $r = 3$ km. Figure 8 shows the graphs of the profile R_1 in each of these four cases, as given by (31), The results show, once again, that the intrinsic ambiguities are very severe. Indeed, the profile R_1 of case 1 diverges from the constant $R_0 = 20$ mm/hr very rapidly. While R_1 in cases 2, 3 and 4 diverges more slowly, the relative error $(R_1 - R_0)/R_0$ dots reach -45 %, -25 % and 37 % respectively already when $r = 2.5$ km. It is also interesting to note that the attenuation factor α is almost identical for the four profiles in cases 3 and 4: an additional attenuation measurement would not be sufficient to distinguish between these two cases.

5 Two beams

How can we attempt to clear up the kind of ambiguity described in the previous two sections? On one hand, one might look for improvements to the physical model of the dependence of the effective reflectivity on the rain rate, as we started to do in the previous section. For example, one might try to impose some conditions on the parameters a, b, α and β (or m, $\overline{D}$ and R), such as bounds on their ranges or functional relations between them which are justified somehow by the physics involved. Such an approach is likely to turn an already quite simplified model into one that is unrealistically constrained. In any case, improved models are likely to involve more ambiguity-producing parameters rather than fewer. The model we have been using, in spite of its simplicity, remains quite useful in understanding the problem at hand. In fact, *because* of its simplicity, the model that assumes power-law $Z - R$ and $k - R$ relationships can be used to test the ability of any improved measurement scheme to discriminate between the ambiguities uncovered so far.

Therefore, rather than modify our simple model, let us consider modifications to our data-gathering scheme. It is apparent from the previous sections that one of the main reasons for the difficulty in finding a unique rain profile given its effective reflectivity profile is that the reflectivity and the accumulated attenuation both contribute to the measured effective reflectivity for any given range, in proportions that are controlled by uncertain parameters. One might therefore look for a measurement scheme that can yield data where the effect of reflectivity on one hand and attenuation on the other are more easily separated. One such approach, somewhat similar to the stereoradar idea proposed by Testud and Amayenc (1989), requires that the radar probe the rain column at two distinct incidence angles, say 0 (nadir-looking) and θ (greater than, but not too far from 0), and that within the volume of the two beams the rain rate is homogeneous horizontally (it may depend on altitude only). This radar viewing configuration is graphically illustrated in figure 9. Unlike the scheme proposed in (Testud and Amayenc 1989), we shall not try to use the effects of the motion of the platform carrying our radar. in fact, we specifically assume that the two radar beam angles are *not* symmetric with respect to any plane normal to the ground - what we do ask of the two beams is that the radar returns which they produce from any given layer have differing attenuations (due to different travel times). For such radar returns to be comparable, we also assume that beam angle θ is sufficiently small so that we may assume that the rain reflectivity and attenuation are essentially independent of the radar beam angle: they can depend on altitude only, not on the horizontal coordinates. For stratiform rain, these are not particularly restrictive assumptions. For highly convective situations, these assumptions might force θ to be too close to 0 to provide any significantly different path-integrated attenuations.

In any case, under these assumptions, the effective reflectivity $p(r)$ measured from range r along the nadir angle is of course still given by (3), but the calibrated effective reflectivity $p_\theta(r)$ from the same range (see figure 9) at angle θ is now given by

$$p_\theta(r) = aR(r\cos\theta)^b 10^{-0.2\alpha\int_0^{r\cos\theta} R(t)^\beta dt/\cos\theta}. \tag{32}$$

Comparing the effective reflectivities along the two incidence angles at the same *altitude* rather than the same range, say at that altitude which corresponds to vertical range r, one finds that if two sets of rainfall parameters $\{R_0(r), a_0, b_0, \alpha_0, \beta_0\}$ and $\{R_1(r), a_1, b_1, \alpha_1, \beta_1\}$ are to produce the same reflectivity levels along both directions, they must satisfy

$$a_1R_1(r)^{b_1}10^{-0.2\alpha_1\int_0^r R_1(t)^{\beta_1}dt} = p(r) = a_0R_0(r)^{b_0}10^{-0.2\alpha_0\int_0^r R_0(t)^{\beta_0}dt}, \tag{33}$$

$$a_1R_1(r)^{b_1}10^{-0.2\alpha_1\int_0^r R_1(t)^{\beta_1}dt/\cos\theta} = p_\theta\left(\frac{r}{\cos\theta}\right) = a_0R_0(r)^{b_0}10^{-0.2\alpha_0\int_0^r R_0(t)^{\beta_0}dt/\cos\theta} \tag{34}$$

at all ranges r. Equating the ratio of the left-hand-sides with that of the right-hand-sides implies that

$$\int_0^r \left(\alpha_0R_0(t)^{\beta_0} - \alpha_1R_1(t)^{\beta_1}\right) dt = 0, \tag{35}$$

and that

$$a_0R_0(r)^{b_0} = a_1R_1(r)^{b_1}, \tag{36}$$

at all ranges r. Therefore we must have

$$R_1 = \left(\frac{\alpha_0}{\alpha_1}\right)^{1/\beta_1} R_0^{\beta_0/\beta_1} = \left(\frac{a_0}{a_1}\right)^{1/b_1} R_0^{b_0/b_1}. \tag{37}$$

Thus, in this case, R_1 and R_0 must be related by a power law. Moreover, documented values for β obtained by several authors (Atlas and Ulbrich 1977, Doviak and Zrnić 1984, Kozu and Nakamura 1991) using statistical regression over C-, X- and K-bands indicate that one can reasonably expect β to be within the interval [0.98, 1.18], which forces the exponent in the power law relating R_0 and R_1 to remain fairly close to 1. Without further constraints, the relative error $|(\alpha_0/\alpha_1)^{1/\beta_1} R_0^{0.1} - 1|$ can still get as large as 1.74, or 174 %, when R_0 exceeds 20 mm/hr, if one uses $\beta_0 = 1.08$, and if $\alpha_0/\alpha_1 = 2$, $\beta_1 = .98$. In this case, however, we *can* essentially eliminate the ambiguity if we have one additional measurement. Specifically, let us assume that we can also measure the average! rain rate R^{avg}. Mathematically, we are assuming that we have an additional equation

$$\int_0^{r_{max}} R_1(r)dr = r_{max} R^{avg} = \int_0^{r_{max}} R_0(r)dr. \tag{38}$$

When we substitute the first expression in (37) for R_1, (38) can be viewed as a formula for determining the linear factor in the power law relating R_1 to R_0, namely

$$\left(\frac{\alpha_0}{\alpha_1}\right)^{1/\beta_1} = \frac{\int_0^{r_{max}} R_0(r)dr}{\int_0^{r_{max}} R_0(r)^{\beta_0/\beta_1} dr}. \tag{39}$$

Thus, under these assumptions, every value of β_1 determines a new profile R_1 uniquely, and unique values for the remaining parameters (the value for α_1 prescribed by (39), and those values for al, b_1 required to satisfy (37)). Specifically, the profile corresponding to β_1 is

$$R_1(r) = \frac{\int_0^{r_{max}} R_0(t)dt}{\int_0^{r_{max}} R_0(t)^{\beta_0/\beta_1} dt} R_0(r)^{\beta_0/\beta_1} \tag{40}$$

Keeping in mind that the ratio β_0/β_1 has to be very close to 1, one can see that any profile R_1 satisfying (40) cannot be very far from the original profile R_0. Let us quantify this ambiguity more precisely: how far off could we be if we tried to use all the data we are assuming is available (i.e. power versus range in two beam angles, plus average rain rate), with $\beta = 1.08$, to solve for the rain rate? We can answer this question by calculating the relative squared error Q between the rain rate R_0 obtained in this fashion (i.e. with $\beta_0 = 1.08$) and any rain rate R_1 that is equally consistent with the data at hand, i.e. any R_1 satisfying 40 for some value β_1. Calling the ratio $1.08/\beta_1 = 1 + \epsilon$, we find that

$$Q(r) = \left(\frac{\int_0^{r_{max}} R_0(t)dt}{\int_0^{r_{max}} R_0(t)^{1+\epsilon} dt} R_0(r)^\epsilon - 1\right)^2. \tag{41}$$

Let us call

$$\begin{aligned} R_{max} &= \sup_r R_0(r), \text{ the maximum value of } R_0, \\ R_{min} &= \inf_r R_0(r), \text{ the minimum value of } R_0. \end{aligned}$$

Assume $\epsilon > 0$: since $R_0(r) \le R_{max}$, we must have $(R_0(r)/R_{max})^{1+\epsilon} \le R_0(r)/R_{max}$, therefore

$$\frac{\int_0^{r_{max}} R_0(t)dt}{\int_0^{r_{max}} R_0(t)^{1+\epsilon}dt} R_0(r)^\epsilon = \frac{\int_0^{r_{max}} R_0(t)/R_{max}dt}{\int_0^{r_{max}} (R_0(t)/R_{max})^{1+\epsilon}dt} (R_0(r)/R_{max})^\epsilon \ge (R_{min}/R_{max})^\epsilon \tag{42}$$

Similarly, since $R_0(r) \geq R_{min}$, we must have $(R_0(r)/R_{min})^{1+\epsilon} \geq R_0(r)/R_{min}$ (ϵ is still positive). Therefore

$$\frac{\int_0^{r_{max}} R_0(t)dt}{\int_0^{r_{max}} R_0(t)^{1+\epsilon}dt} R_0(r)^\epsilon = \frac{\int_0^{r_{max}} R_0(t)/R_{min}dt}{\int_0^{r_{max}} (R_0(t)/R_{min})^{1+\epsilon}dt} (R_0(r)/R_{min})^\epsilon \leq (R_{max}/R_{min})^\epsilon. \quad (43)$$

Putting (42) and (43) together, we find that if $\epsilon > 0$

$$Q(r) \leq \text{the greater of } \left(\left(\frac{R_{min}}{R_{max}}\right)^\epsilon - 1\right)^2 \text{ or } \left(\left(\frac{R_{max}}{R_{min}}\right)^\epsilon - 1\right)^2. \quad (44)$$

The same inequality can be derived using similar considerations in the case where ϵ is negative.

We can usc *(44)* to get an upper bound on the worst-case relative squared error Q, as a function of the ratio R_{max}/R_{min}, with ϵ = O. 1 (th e worst case, according to regression estimates of β mentioned above). Figure 10 shows a plot of $\sqrt{Q}$ versus R_{max}/R_{min}. As one can SCC from the graph, as long as $R_{max}/R_{min} < 5$ (a very realistic bound on the variation of the rain rate within a single '<event"), the relative error cannot, exceed 1 7% at any range. Thus, in this case, in spite of the fact that the theoretical solutions are not unique, this ambiguity is acceptable because the difference between the solutions is never too large. of Course, the uncertainty in the estimate of R^{avg} that one would have in practice would have to be taken into account by any inversion algorithm.

There remains to verify that the tilt angle θ does not need to be prohibitively small. Working backwards, wc usc (32) to note that the difference between the echo powers along the nadir and tilted look angles is 10 $\log_{10}(1/\cos \theta)$. Using Rayleigh fatling as our benchmark noise level, if we usc 50 effectively independent radar pulses to obtain our data, the r. m. s. noise level in our data will be $1/\sqrt{50}$, or about 0.14 dB. Therefore, for our two-angle power difference to exceed this noise floor, it is sufficient to choose θ sue]) that $10 \log_{10}(1/\cos \theta) \geq$ 0.14, i.e. $\theta \geq 14.4°$. From the geometry of figure 9, it is now easy to translate this into the corresponding horizontal homogeneity requirement. Specifically, if ϕ is the beamwidth of the radar antenna, h the altitude of the tallest rain cell, and D the altitude of the platform, we must assume horizontal homogeneity over a distance of $x(\theta) \simeq (h \sin \theta + 2D\phi)/\cos \theta$. With $\phi = 0.15°$, $D = 300$ km, $h =$ IO km and $\theta = 15°$, we find that $x(15°) \simeq 4.3$ km, a quite reasonable lower bound on the horizontal resolution of this scheme. Finall y, since the two radar looks Cover roughly the same footprint on the ground, the corresponding data takes will have to occur at slightly different points in time, and we need to verify that the rain cel] cannot change much in the intervening time. At an altitude of about 300 kin, and assuming an approximate speed of I O I{ n/sec, the platform would need to travel for 300 tan(1 50)/1 () $\simeq$ 8 seconds between the two data takes. Assuming an approximate fall velocity of $3.8 D^{0.67}$

11/sec, drops with a diameter as large as $D = 5$ mm will not travel farther than 90 meters in the intervening time, or less than a typical radar range resolution bin. Thus, we conclude that the two-beam approach which we propose is quite feasible, and should produce largely unambiguous results.

6 Two frequencies

It is natural to expect that if we could analyze the back-scattered power at two distinct frequencies, the problem of estimating the rain profile would be significantly less ambiguous. Indeed, if we approximated the continuous rain rate function $R(r)$ by a piecewise constant version $R(n\Delta r)$, $1 \leq n \leq N$ by choosing some arbitrary constant "layer thickness" Δr, we would then have $N + 8$ unknowns (the additional eight unknowns correspond to the parameters a, b, α, β at the two frequencies, assumed constant) on one hand, and $2N$ equations identical to (3) (one for every discrete layer at each of the two frequencies) on the other hand. Thus, as soon as $N \geq 8$, one would be able to solve for the rain profile unambiguously. of course, for that to be true, one would have to make sure that the $2N$ equations are sufficiently independent. How can we find out if this is the case?

The two-beam scenario of the last section is one example where, had we chosen to discretize the atmosphere into N layers, we would have ended up with $2N$ equations (namely, (3) and (32) for each layer) and only $N + 4$ unknowns, yet, as we demonstrated, the equations would have admitted multiple solutions. The simplest two-frequency example confirms that this can indeed happen in the case at hand too: indeed, if the rain rate is constant, many rain rate profiles (with different a, b, α, β values) can theoretically be responsible for the same received powers at both frequencies.

To study the problem more systematically, we proceed as in the previous section. Assume that R_1 and R_0 satisfy the equations

$$a_1 R_1(r)^{b_1} 10^{-0.2\alpha_1 \int_0^r R_1(t)^{\beta_1} dt} = a_0 R_0(r)^{b_0} 10^{-0.2\alpha_0 \int_0^r R_0(t)^{\beta_0} dt} \quad \text{(frequency 1)}, \tag{45}$$

$$c_1 R_1(r)^{d_1} 10^{-0.2\gamma_1 \int_0^r R_1(t)^{\delta_1} dt} = c_0 R_0(r)^{d_0} 10^{-0.2\gamma_0 \int_0^r R_0(t)^{\delta_0} dt} \quad \text{(frequency 2)}, \tag{46}$$

at all ranges r. As in the single-frequency case, if we replace R_1 by the new unknown $F = R_1 / R_0$, we can transform these equations into

$$\frac{F'}{F} = \frac{0.2 \log(10)\alpha_1}{b_1} R_0^{\beta_1} F^{\beta_1} + \left(\frac{b_0}{b_1} - 1\right)\frac{R'_0}{R_0} - \frac{0.2 \log(10)\alpha_0}{b_1} R_0^{\beta_0}, \tag{47}$$

$$\frac{F'}{F} = \frac{0.2\log(10)\gamma_1}{d_1} R_0^{\delta_1} F^{\delta_1} + (\frac{d_0}{d_1} - 1)\frac{R'_0}{R_0} - \frac{0.2\log(10)\gamma_0}{d_1} R_0^{\delta_0}, \tag{48}$$

together with the initial condition $F(0) = (a_0/a_1)^{1/b_1} R_0(0)^{b_0/b_1 - 1} = (c_0/c_1)^{1/d_1} R_0(0)^{d_0/d_1 - 1}$. By eliminating F' between (47) and (48), we can obtain an equation for $F(r)$ at each range r, namely

$$F^{\beta_1} - \frac{@}{d_1}\frac{}{\alpha_1} R_{n_0}^{\delta_1 - \beta_1} F^{\delta_1} = \frac{5b_1}{\alpha_1 \log 10}\left(\frac{d_0}{d_1} - \frac{b_0}{b_1}\right)\frac{R'_0}{R_0^{1+\beta_1}} + \frac{b_1}{\alpha_1}\left(\frac{\alpha_0}{b_1} R_0^{\beta_0 - \beta_1} - \frac{\gamma_0}{d_1} R_0^{\delta_0 - \beta_1}\right) \tag{49}$$

We shall use the inverse function theorem (IFT) to show that this equation can be solved for $F(r)$ at any particular r. For the IFT to apply, we need to make sure that the left-hand-side of (49) is an invertible function of F, at least for those values of the parameters β_1, δ_1 that we are interested in, namely values that are very close to 1. To that end, we compute the derivative of the function $f(x) = x^\beta - \lambda x^\delta$: $f'(x) = \beta x^{\beta-1} - \lambda\delta x^{\delta-1}$, and, as long as $\lambda \neq \beta/\delta$, $f'(x)$ is indeed non-zero. Since f is smooth in β, δ, λ, it must remain generically locally invertible for values of β and δ close to 1. Thus, by the inverse function theorem, the left hand-side of 49 is generically an invertible function of the quantity $F(r)$. Therefore, generically, given constant values for the parameters a, b, c, d, $\alpha, \beta, \gamma, \delta$, and once the original profile R_0 is specified, (49) admits at most one solution in $F(r)$. The calculations necessary to obtain F explicitly are quite tedious and unenlightening. Symbolically, we can write the solution as

$$F(r) = \mathcal{F}(R_0(r), R'_0(r), \text{a]}, b_1, \alpha_1, \beta_1, c_1, d_1, \gamma_1, \delta_1, a_0, b_0, \alpha_0, \beta_0, c_0, d_0, \gamma_0, \text{fro)}, \tag{50}$$

where the function $\mathcal{F}$ is smooth in all its arguments, including the first two. In turn, for such a, solution F to satisfy (47) and (48), we would need to substitute the expression (50) for F in either *(47)* or (48), differentiate $\mathcal{F}$ as required, and make sure that the resulting equation is satisfied. But, treating $a_1, b_1, \alpha_1, \beta_1, c_1, d_1, \gamma_1, \delta_1, a_0, b_0, \alpha_0, \beta_0, c_0, d_0,$ 70,6'0 as parameters, the resulting equation is a non-linear second-order ordinary differential equation in R_0. Its solutions can be parametrized by at most 18 parameters (chosen from among the 16 already at hand, together with the two initial values for R_0 and R'_0). A rain rate profile R_0 which dots not belong to this family of solutions will automatically be an unambiguous profile: no values of a, b, α, β, c, d, γ, δ can produce a different profile that would nevertheless generate the same backscattered power at two frequencies. 011 the other hand, a profile R_0 that dots belong; to this family of solutions (such as, for example, any exactly constant profile, as we pointed out earlier) remains ambiguous even with dual-frequency measurements. Still, since the family of ambiguous profiles is thus parametrized by a finite number of parameters, one can assert that, *generically*, the two-frequency problem has a unique solution.

7 Conclusions

The formulas derived above confirm that estimating rain using a dual-frequency air- or space-borne radar is generically unambiguous, whereas a single-frequency single-look-angle system has substantial inherent ambiguities which additional data, such as])ath-averaged attenuation, can somewhat lessen but certainly not remove. The formulas which we derived assuming power-]aw $Z - R$ and $k - R$ relations, as well as the formulas based on a drop-size-distribution model, arc useful for study ing specific cases, as well as for assessing the effect of known uncertainties in certain measured parameters (such as the surface backscattering coefficient, in the case of surface-referenced measurements) on the achievable accuracy of the rain rate profile retrieval. The examples which wc presented (figures 1-6 illustrating 6 instances of the ambiguities in the direct profiling approach assuming power-law Z - R and $k - R$ relationships, figure 7 illustrating in effect 4 instances of the ambiguities when the surface-rcfcrcncc approach with power-law Z - R and $k - R$ relationships is used, and figure 8 illustrating 4 cases of the ambiguities when the direct approach is used with Z - R and k -- R relationships based on the drop size distribution) show that the single-frequency ambiguities exceed 100 % in general, and can easily exceed 40 % even when the path-integrated attenuation is assumed given.

If one frequency is all that is available, our two-stall-allglcs approach dots reduce the am biguities to a point where they arc scientifically insignificant, once the path- averaged rail]-rate is also known. in this case, the ambiguities should typically not exceed 20%.

For systems such as TRMM, cmc must try to make the best estimate using the single-beam, single-frequency radar data, eventually with the help of radiometer measurements. The results presented above highlight the need for further work in three areas:

1) A detailed study (empirical and physical) of the *interdependence* of the parameters governing the Z -- R and k -- R relations (whether one uses power-law approximations or relations derived from the drol)-size distribution), and, as a second step, of the constraints that govern their evolution with altitude. This would allow one to place tighter constraints on the parameters in question, and correspondingly reduce the ambiguity which they cause.

2) Developing algorithms that can incorporate any (remaining) uncertainties in the Z - R and k -- R parameters, and produce estimates that can account for these uncertain ties in the retrieved raill-rate profile. Such algorithms would then quantify the resulting uncertainty by enabling one to calculate the *variance* of the estimates along with the estimated rairl-rate values themselves.

3) Developing an approach that can fuse the data obtained using a single-frequency radar

and a single- or multi-channel radiometer, and derive the best estimate for the rain profile given these two sets of data.

A preliminary summary of our progress in addressing the last two issues can be found in (Haddad and 1m 1993). The details of our approach and further results will be reported in future publications.

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Figure captions

Figure I: Constant rain rate $R_0 = 40$ mm/hr and $\alpha_0 = 0.036$, and ambiguous profile R_1 with $\alpha_1 = 0.018$.

Figure 2: Constant rain rate $R_0 =$ '20 mm/hr and $\alpha_0 = 0.018$, and ambiguous profile R_1 with $\alpha_1 = 0.036$.

Figure 3: Rain rate profiles R_0 with $\alpha_0 = 0.018$, and R_1 with $\alpha_1 = 0.036$.

Figure 4: Rain rate profiles R_0 with $\alpha_0 = 0.036$, and R_1 with $\alpha_1 = 0.018$.

Figure 5:]1.sill rate profile R_0 as in figure 4 with $b_0 = 1.24$, α_0: 0.018, $\beta_0 = 0.98$, and R_1 with $a_1 = a_0/0.73$, $b_1 = 1.8$, $\alpha_1 = 0.03($; ,& $= 1.18$.

Figure 6: Rain rate profile R_0 as in figure 4 with $b_0 = 1.34$, $\alpha_0 = 0.018$, $\beta_0 = 0.98$, and R_1 with $a_1 = a_0/1.03$, $b_1 = 1.64$, $\alpha_1 = 0.036$, /?1 $= 1.18$.

Figure 7: Constant rain rate $R_0 = 20$ mm/hr, with two examples of profiles that are ambiguous with R_0 in the surface-rcfcrcncc approach.

Figure 8: Constant rain rate $R_0 = 20$ mm/hr, with four examples of profiles that are ambiguous with R_0 in the dlo]~-size-clistliblltioll approach.

Figure 9: '1'wo-look-angle profiling scheme.

Figure 10: upper bound for the relative error in the two- look-angle approach.

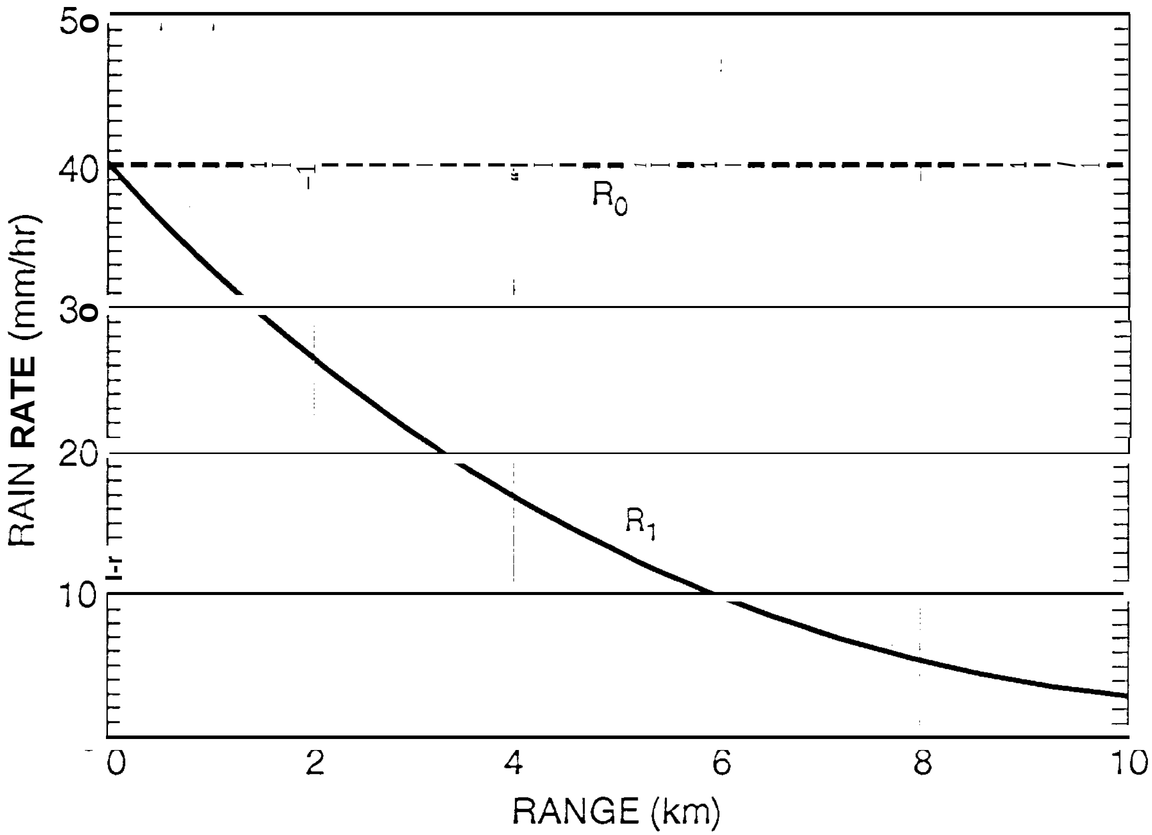


Figure 1.

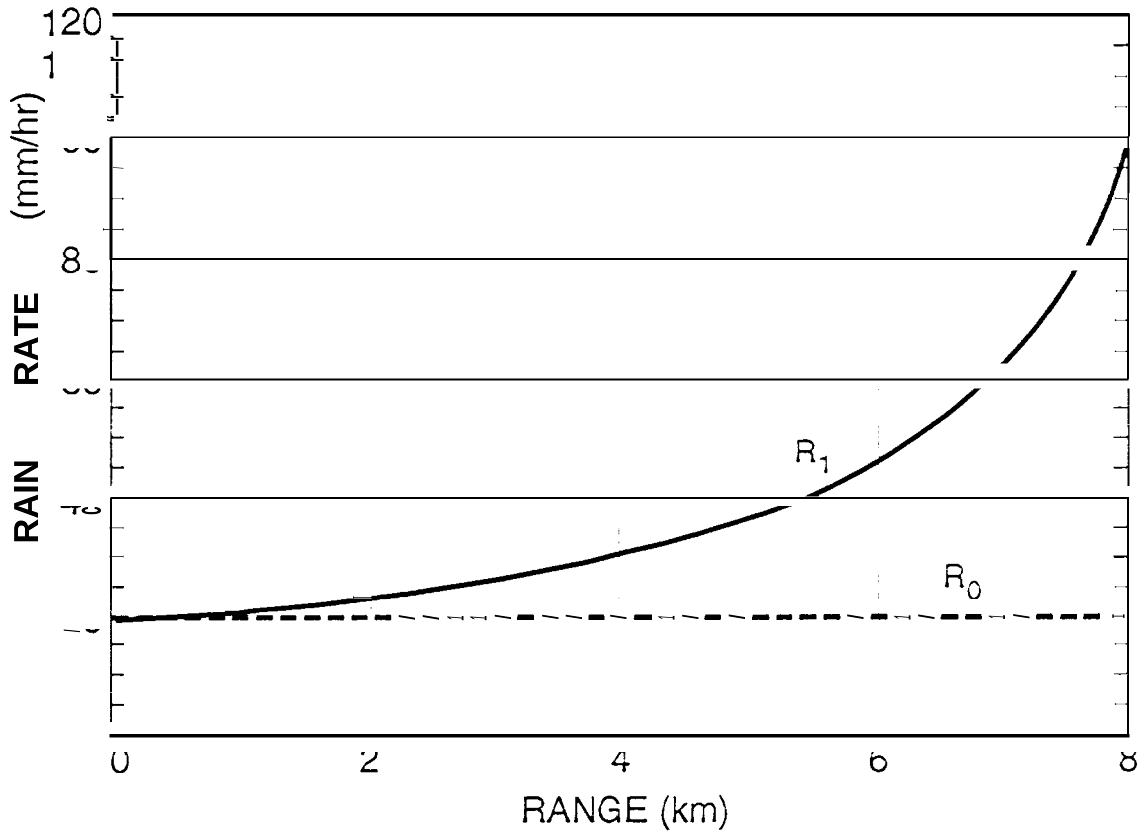


Figure 2.

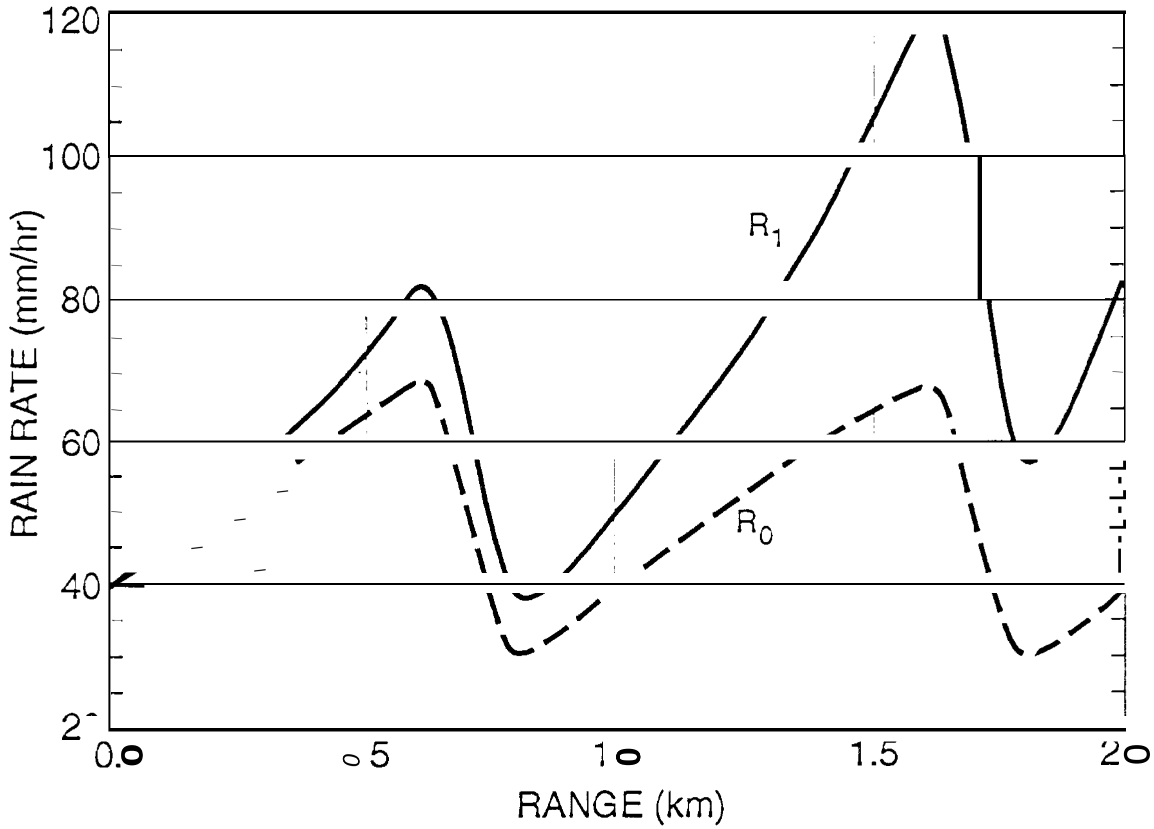


Figure 3.

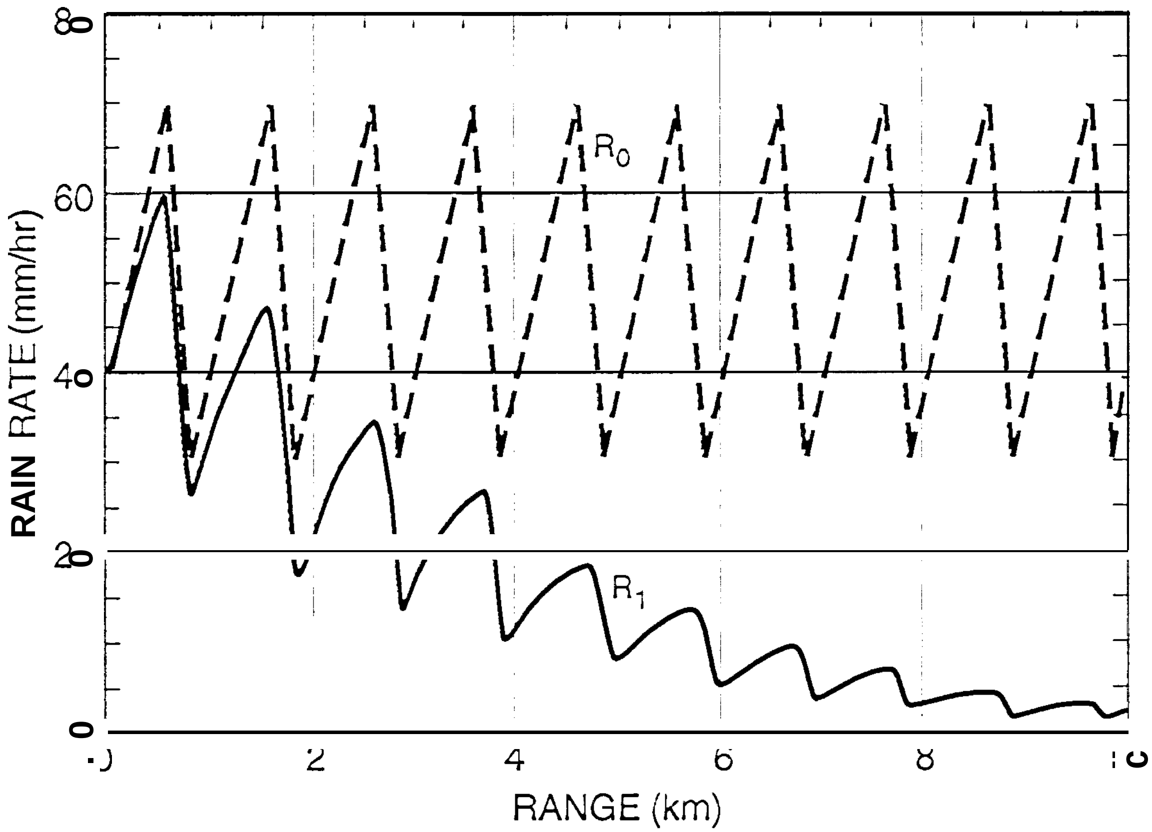


Figure 4.

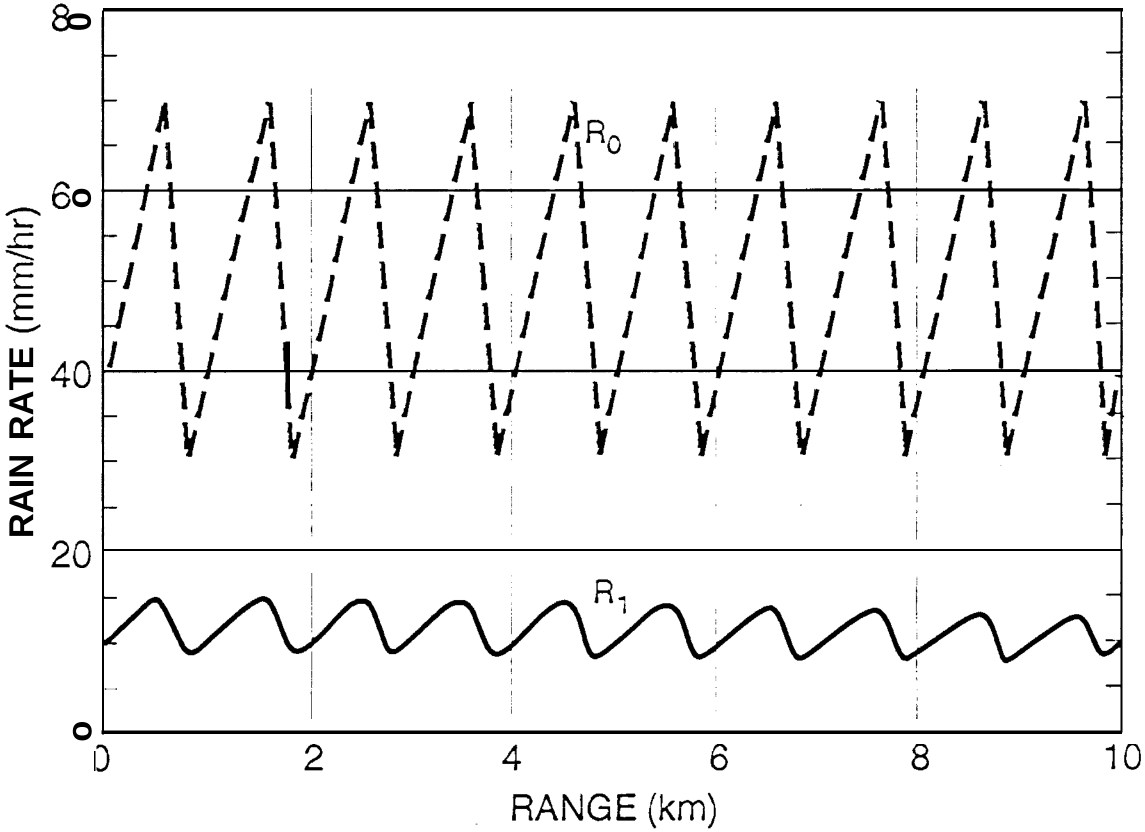


Figure 5.

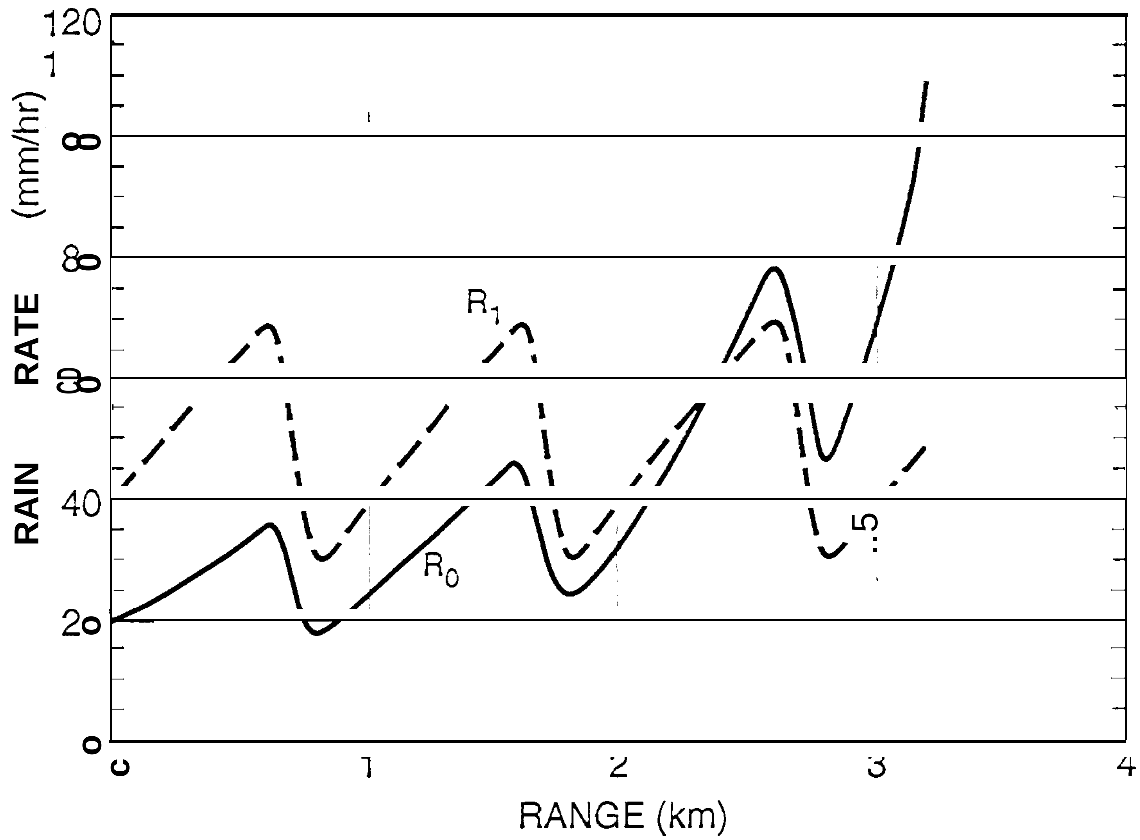
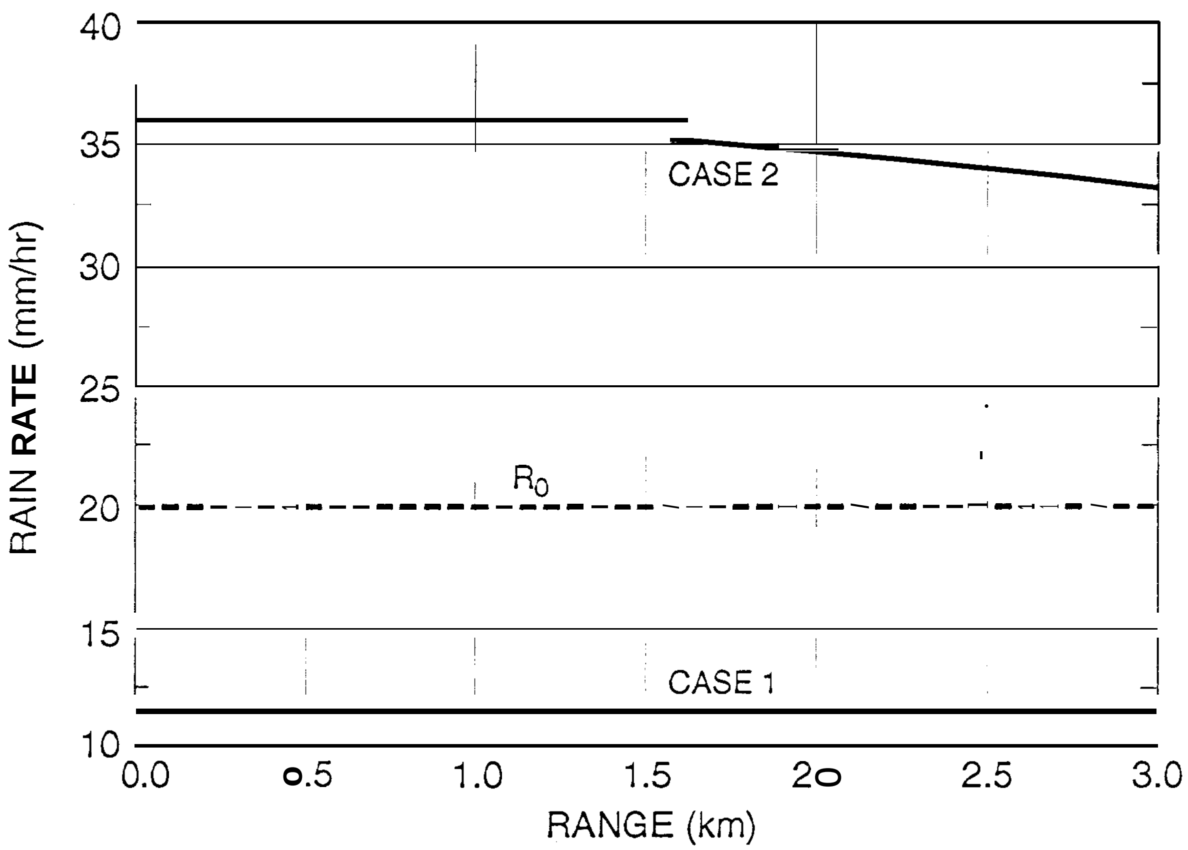
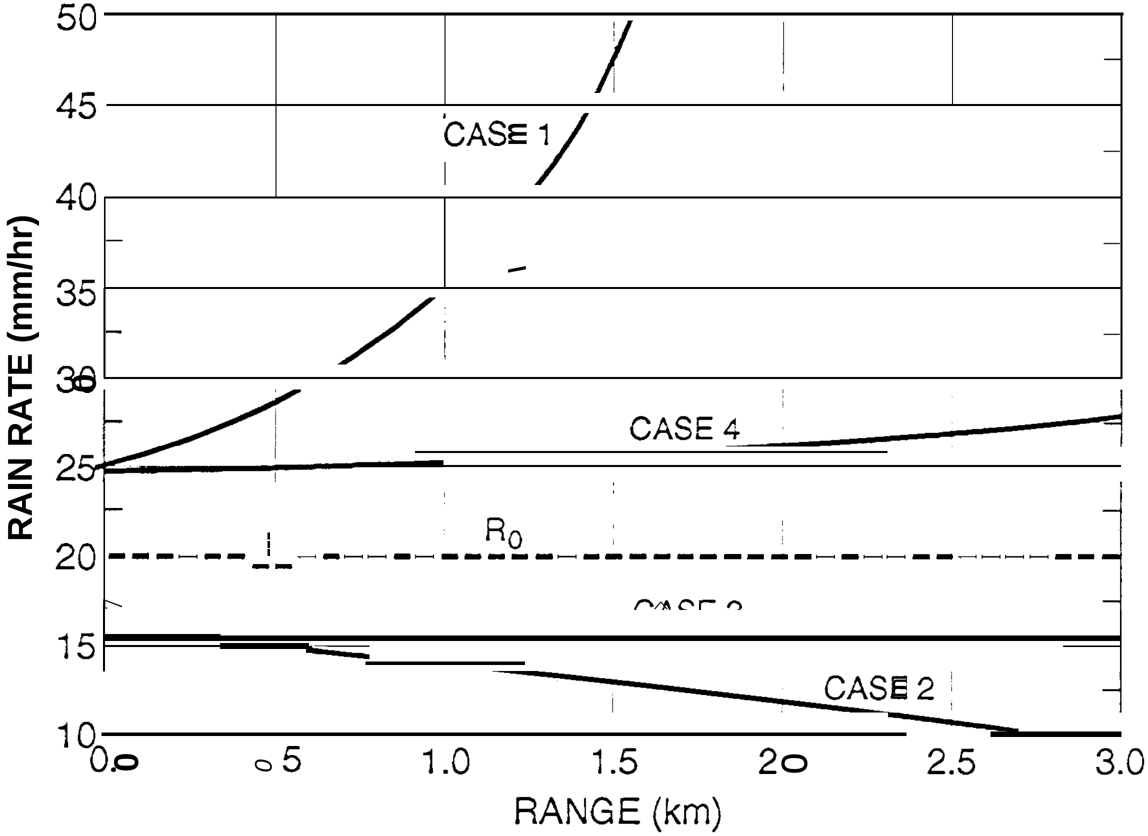


Figure 6.



RAIN RATE (mm/hr)
40
35
30
25
20
15
10
CASE 2
R_0
CASE 1
0.0
0.5
1.0
1.5
2.0
2.5
3.0
RANGE (km)

Figure 7.



RAIN RATE (mm/hr)
50
45
40
35
30
25
20
15
10
CASE 1
CASE 4
R_0
CASE 2
0.0
0.5
1.0
1.5
2.0
2.5
3.0
RANGE (km)

Figure 8.

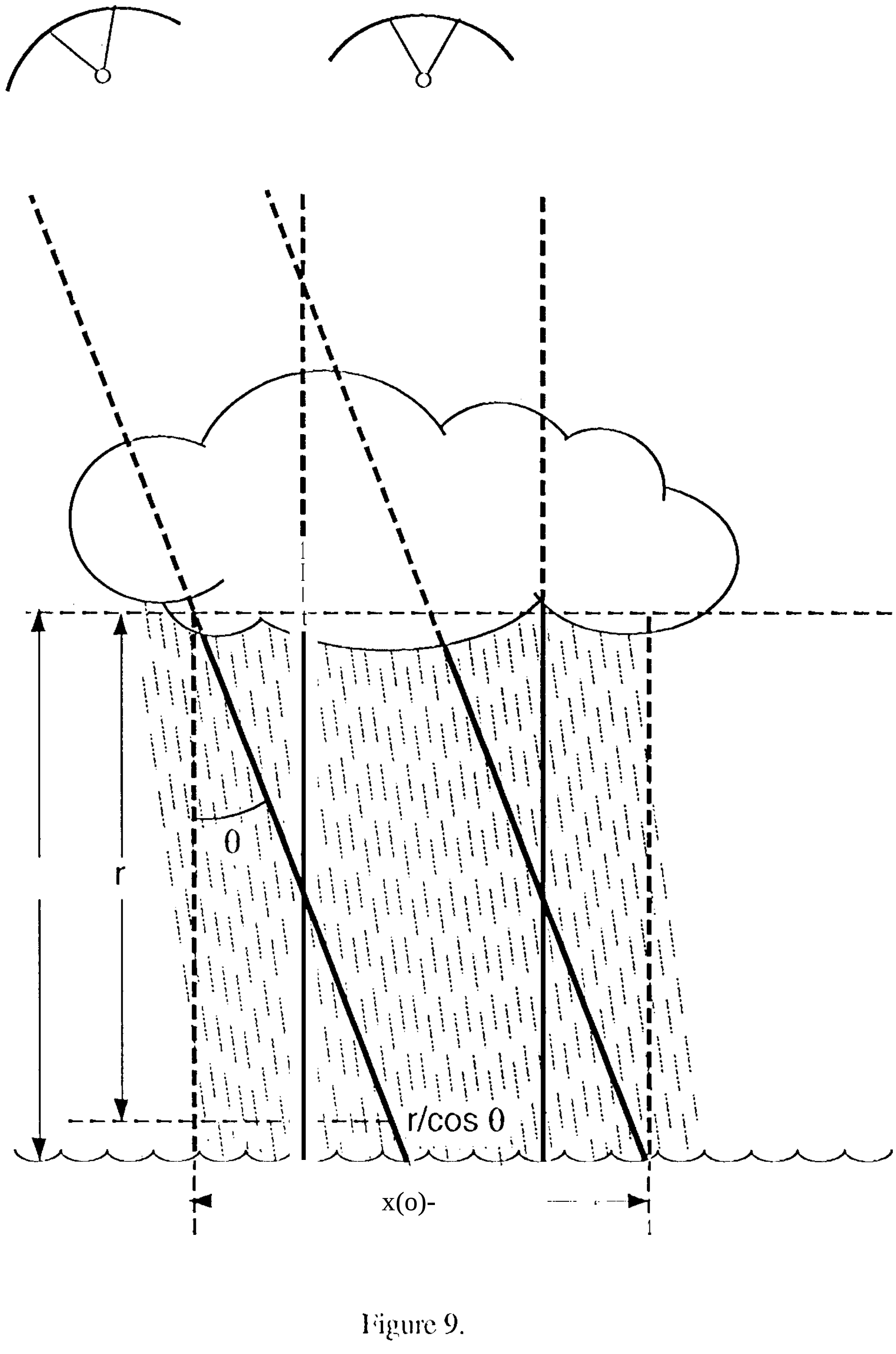


Figure 9.

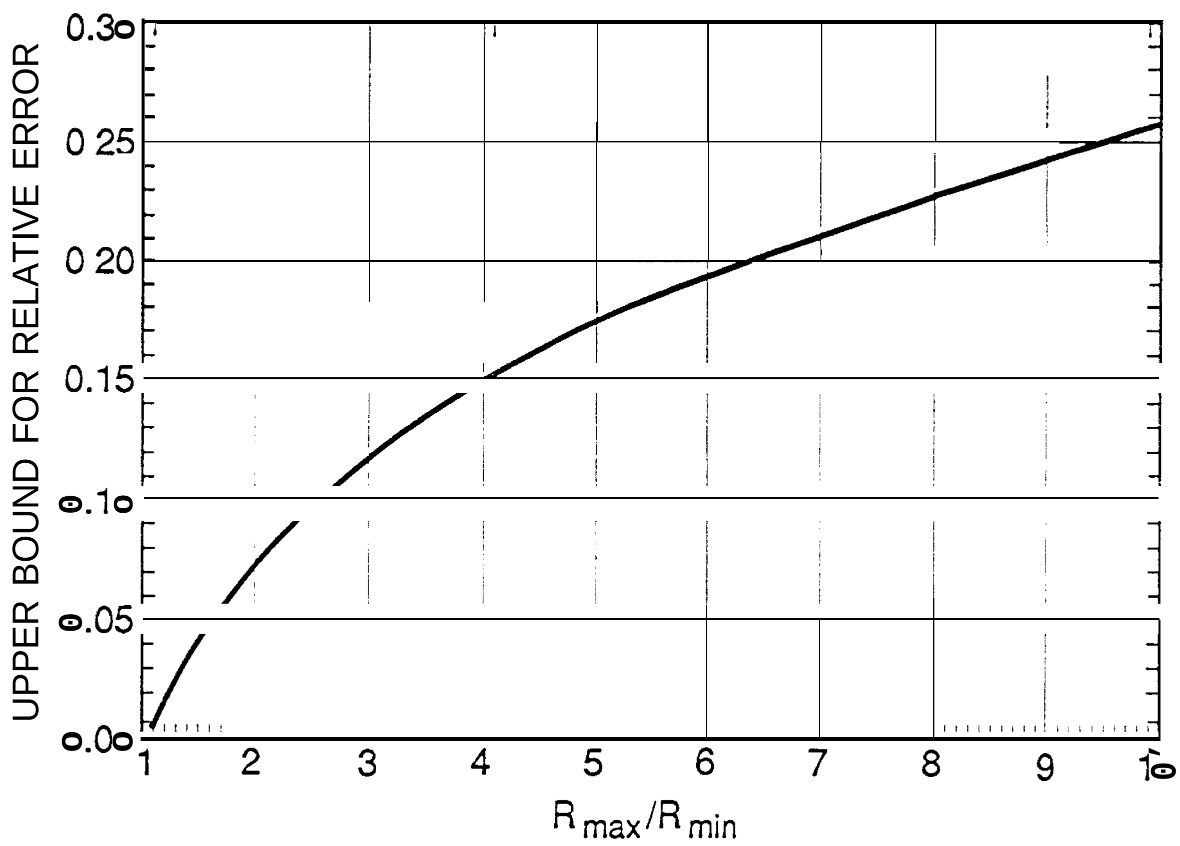

UPPER BOUND FOR RELATIVE ERROR
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1
2
3
4
5
6
7
8
9
10
R_{max}/R_{min}

Figure 10.