ABSTRACT

OPTIMAL LOW THRUST TRAJECTORIES USING DIFFERENTIAL INCLUSION CONCEPTS

by

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submitted to the

AIAA/AAS SPACE FLIGHT MECHANICS MEETING
COCOA BEACH, FLORIDA

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Introduction

Low thrust propulsion systems typically have their greatest benefit for high energy missions or missions with large post-launch maneuver requirements. Missions which have been examined include main belt asteroid rendezvous, comet rendezvous, outer planet and Mercury orbiters, Pluto flyby and solar probe missions.[1] Low thrust mission design software used to determine these trajectories is based on two distinct formulations of the optimal control problem: the indirect and direct methods. The traditional approach (indirect) is to use the calculus of variations to obtain first-order necessary conditions on the states and costates. These methods are referred to as indirect because an extremal to the cost functional is obtained by satisfying the first-order necessary conditions, Indirect methods typically result in solving two-point boundary value problems that require integration of the state and costate differential equations. In contrast, direct methods are conceptually different in that no explicit integration takes place. Rather, a finite approximation is sought using finite difference methods. Direct methods often transform the optimal control problem into a nonlinear programming problem (NLP). The cost functional is then directly minimized by varying discrete values for the states and controls. A direct method based on differential inclusion concepts has been developed and successfully used to compute low thrust trajectories. This new approach removes explicit control dependence from the problem thereby reducing the dimension of the parameter space for the NLP. Also when compared to other direct methods, fewer nonlinear constraints are required to represent the dynamics of the problem.

Problem Statement

The equation of motion for a spacecraft subject to a single gravitational source is given through the rocket equation:

$$\ddot{\mathbf{r}} = \mathbf{g}(\mathbf{r}) + \frac{T}{m}\hat{\mathbf{u}} = \mathbf{g}(\mathbf{r}) + \Gamma \tag{1}$$

where **r is** the position vector, **g(r)** the gravitational acceleration vector, T the engine thrust, $\hat{\mathbf{u}}$ a unit vector in the thrust direction, Γ the thrust acceleration vector and m the mass of the vehicle. The model used for the spacecraft's propulsion system, whether constant specific impulse (csi) or variable specific impulse (vsi), directly affects the relationship between control effort and propellant consumption, Assuming a constant power source P,

such as nuclear electric, the equations that govern the change in mass for both csi and vsi systems are given below. The variable c is the propellant's exhaust velocity.

csi case: $$m = -\frac{T}{c} \tag{2}$$

vsi case: $$\frac{d}{dt}\left(\frac{1}{m}\right) = \frac{\Gamma^2}{2P} \tag{3}$$

For convenience, a state vector x will be defined. For the csi case, $\mathbf{x}^T = [\,\mathbf{r}\ \mathbf{v}\ \gamma\,]$ where v is the velocity vector and γ = in (m). For the vsi case, XT = [$\mathbf{r}\ \mathbf{v}\ \alpha$] **where** $\alpha = \frac{P}{m}$. The state rates for the two systems can then be obtained through equations (l)-(3).

$$\dot{\mathbf{r}} = \mathbf{v} \tag{4}$$

csi case: $$\mathbf{v} = \mathbf{g}(\mathbf{r}) + T \exp(-\gamma)\,\hat{\mathbf{u}} \tag{5}$$

$$1' = -\frac{T \exp(-\gamma)}{c} \tag{6}$$

$$\dot{\mathbf{r}} = \mathbf{v} \tag{7}$$

vsi case: $$\dot{\mathbf{v}} = \mathbf{g}(\mathbf{r}) + \Gamma \tag{8}$$

$$\alpha = \frac{\mathbf{r}^T\mathbf{r}}{2} \tag{9}$$

The set of differential equations given in equations (4)-(6) can be written as $x = \mathbf{f}_{csi}(\mathbf{x}, T\hat{\mathbf{u}})$ and equations (7)-(9) as $x = \mathbf{f}_{vsi}(\mathbf{x}, \Gamma)$. To determine an optimal trajectory, controls ($T\hat{\mathbf{u}}$ or Γ) must be chosen to satisfy any boundary conditions on the states while minimizing an objective function.

Recently, it has been shown that many optimal control problems can be described by functional differential inclusions.[2-5] Differential inclusions represent the dynamics of a problem in terms of attainable sets rather than differential equations. Seywald[5] was first to show how differential inclusion could be used to solve one-dimensional trajectory optimization problems. Examination of equations (4)-(6) and (7)-(9) show that γ and α contain information about the control magnitude but not the control direction. To remove explicit dependence of the differential equations on the control, the v equations are manipulated to produce a scalar equation of the form:

csi case: $$(\dot{\mathbf{v}} - \mathbf{g}(\mathbf{r}))'''\,(\dot{\mathbf{v}} - \mathbf{g}(\mathbf{r})) = \gamma^2 c^2 \tag{lo}$$

vsi case: $$(\dot{\mathbf{v}} - \mathbf{g}(\mathbf{r}))^{T} (\dot{\mathbf{v}} - \mathbf{g}(\mathbf{r})) = 2\,\dot{\alpha} \tag{11}$$

This manipulation has replaced the control variables with state rates. Using a finite difference approach, the total maneuver time is divided into N segments. Define the end points of each segment as the left and right nodes denoted by subscripts 1 and r and assume equal segment lengths t_s. If a first-order approximation is used for the derivative across each segment, then the state rates may be represented as:

$$\dot{\mathbf{r}} \cong \frac{\mathbf{r}_r - \mathbf{r}_l}{t_s} \qquad \dot{\mathbf{v}} \cong \frac{\mathbf{v}_r - \mathbf{v}_l}{t_s} \qquad \dot{\gamma} \cong \frac{\gamma_r - \gamma_l}{t_s} \qquad \text{and} \qquad \dot{\alpha} \cong \frac{\alpha_r - \alpha_l}{t_s} \tag{12}$$

Substituting the position derivative approximation given in equation (12) into equations (4) and **(7)** and evaluating the equation at the segment center yields 3N linear equality constraints. Assuming fixed time t_f maneuvers, $t_s = t_f/N$.

$$\mathbf{r}_r - \mathbf{r}_l - \frac{(\mathbf{v}_r + \mathbf{v}_l)t_s}{2} = 0 \tag{13}$$

The scalar constraint on the velocity state rate evaluated at the segment center becomes N nonlinear equality constraints.

csi case: $$(\mathbf{v}_r - \mathbf{v}_l - t_s\mathbf{g}(\mathbf{r}_c))^{T}(\mathbf{v}_r - \mathbf{v}_l - t_s\mathbf{g}(\mathbf{r}_c)) - 2c^2(\gamma_r - \gamma_l)^2 = 0 \tag{14}$$

vsi case: $$(\mathbf{v}_r - \mathbf{v}_l - t_s\mathbf{g}(\mathbf{r}_c))^{T}(\mathbf{v}_r - \mathbf{v}_l - t_s\mathbf{g}(\mathbf{r}_c)) - 2t_s(\alpha_r - \alpha_l) = 0 \tag{15}$$

where $\mathbf{r}_c = \dfrac{\mathbf{r}_l + \mathbf{r}_r}{2}$

The mass related variables evolve subject to the following N nonlinear and linear inequality constraints. Note that this method does not need to assume a control structure for the csi case. The structure (burn-coast-burn, bum) is contained in they values.

csi case: $$0 \le -(\gamma_r - \gamma_l)\exp\left(\frac{\gamma_r + \gamma_l}{2}\right) c \le T\,t_s \tag{16}$$

csi case: $$0 \ge \gamma_r - \gamma_l \tag{17}$$

vsi case: o s c x r - a * s - (18)

Results

A variety of trajectories have been solved using the above direct method. However, due to space limitations only two trajectories will be shown. The first example is taken from reference [6] and is a two-dimensional gravity-free, csi case that involves a maximum velocity transfer to a rectilinear path. The problem statement is to transfer a particle using a thrust magnitude T with a thrust direction angle $\beta(t)$ from rest at the origin to a path parallel to the x-axis a distance h away in a given time t_f arriving with zero velocity in the y direction and maximum velocity in the x direction. Figures 1-4 display some characteristics of the optimal trajectory. The following conditions were placed on the transfer. $m(t_0)=1$, T=l, $t_f=1$, c=l, N=10 and h=0.1. The entire mass of the particle was assumed to be propellant. Cases were also run with a limit on the final mass and, the optimal solution was to use all available propellant. Figure 4 contains two curves for the control angle β. As described in [6], indirect methods may be employed to show that the optimal control angle is given by the bilinear tangent law given by equation (19). The values of constants c_1= 3.38* 10-5, $c_2=-3.68*10^{-2}$, $c_3=-1.00*10^{-1}$, and $c_4=-2.87*10^{-2}$ were obtained through information on the initial costates. Values are plotted at the segment endpoints.

$$\tan(\beta) = \frac{-c_2 t + c_4}{-c_1 t + c_3} \tag{19}$$

The second curve comes from post-processing the output from the NLP and is plotted at the segment centers. If the approximations for the velocity rate of change shown in equation (12) are used, the control angle can also be calculated through the equation $\beta = \tan^{-1}(v_y/v_x)$. The two curves, one supplied through an indirect method and the other through a direct method, very closely match.

The second example is an interplanetary transfer using a vsi propulsion system. The trajectory shown is a 450 day Earth to Mercury transfer with 32 segments. Launch date was chosen to be December 17, 1999. Trajectories with intermediate bodies have also been calculated. The modeling of the gravity assists and supporting figures will be contained in the full paper.

In summary, this paper discusses a technique for calculating optimal low-thrust trajectories using a finite dimensional approximation to the continuous time problem, The technique formulates a NLP where the control parameters are replaced with state rate

information thereby reducing the parameter space of the resulting NLP. The described method exhibits good convergence from a variety of initial starting solutions.

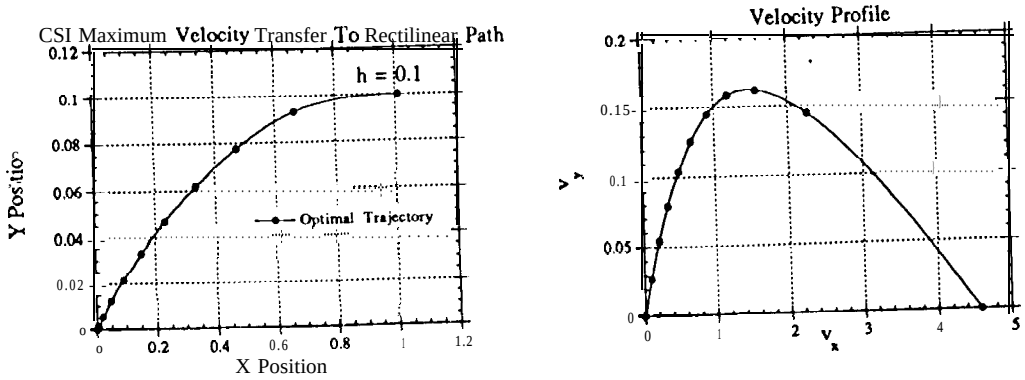

Figure 1-2: Position and Velocity Profiles

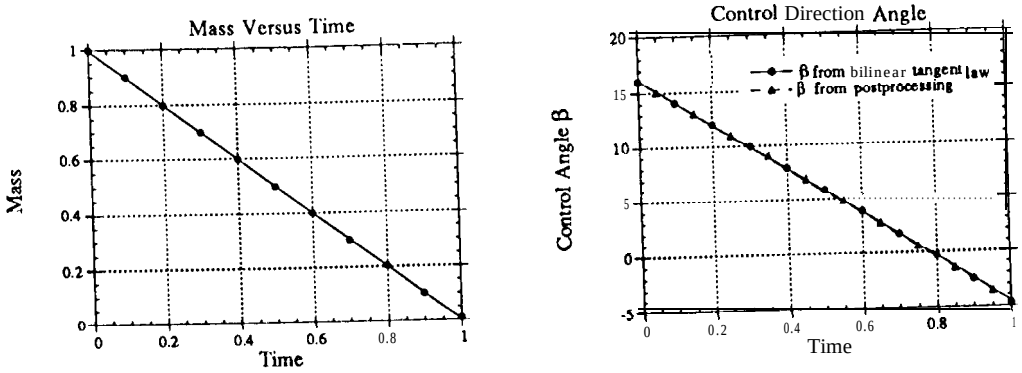

Figure 3-4: Mass and Control Angle Versus Maneuver Time

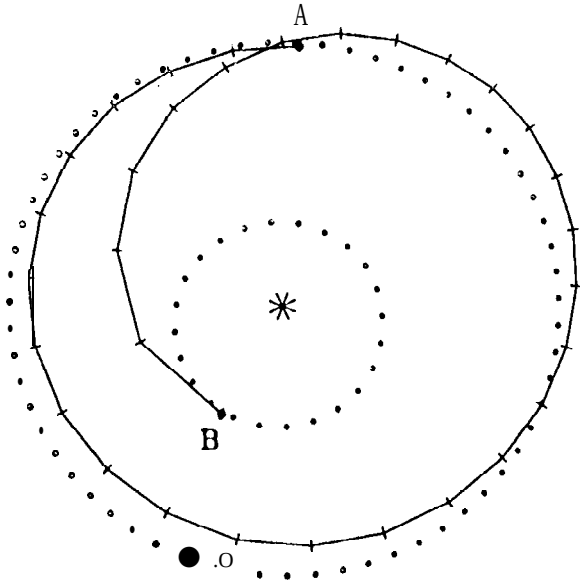

Figure 5. Earth-Mercury Variable Specific Impulse Trajectory

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