

A Synthesis Approach to the Design of Oversampled Data Converters

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Abstract

A synthesis approach to the generation of a quantized sequence based on an oversampled input sequence is presented. The generation algorithm is chosen to minimize a metric that measures the amount of error power that resides in the bandwidth occupied by the desired signal. The first-order $\Delta\Sigma$ modulator is a special case of the synthesis approach when the input is constant. The synthesis approach has superior in-band noise performance over the first-order $\Delta\Sigma$ modulator for finite oversampling ratios, and superior in-band noise performance over conventional $\Delta\Sigma$ converters of arbitrary order when the oversampling ratio is less than 2.862.

1 Introduction

Single-bit data converters are attractive because of their guaranteed linearity and their simple analog circuitry. Recent research in the field of Direct Digital Synthesis (DDS) has shown the need for high-linearity, high-speed digital-to-analog converters (DACs) that do not necessarily have high resolution [1], in addition, such systems often generate signals that span large bandwidths relative to state-of-the-art digital clock rates. Therefore, the use of $\Delta\Sigma$ modulators [2] is often ruled out due to the low oversampling ratios that can be accommodated. The work presented here treats quantizers with an arbitrary number of bits, so the single-bit converter is a special case. In addition, this work takes into account potentially low oversampling ratios.

The present effort departs from the standard analytic approach to the design of oversampled data converters. In most of the literature, an architecture is first proposed and subsequently analyzed. More recent alternatives to conventional $\Delta\Sigma$ architectures [3,4], while effective, often lack explicit theoretical motivation, and one naturally wonders if a given architecture is optimal in any sense. The new approach presented here is best described as synthetic: a performance metric based on spectral error characteristics is proposed and a generation algorithm is then chosen to minimize the performance metric.

2 A Performance Metric and the Synthesis Algorithm

It is desired to generate a quantized sequence, $y[n]$, based on an input sequence, $x[n]$. The values of $y[n]$ are to be chosen from an arbitrary quantized set. In the binary case, $y[n]$ is either $+c$ or $-c$, where c is a constant. The output error sequence is defined as $e[n] = y[n] - x[n]$. Our goal is to select the output sequence that minimizes the error power in the frequency region $|\omega| < \pi/R$, where R is the oversampling ratio. The approach presented here sequentially chooses the values of $y[n]$ that minimize the following time-dependent performance metric:

$$\xi_n = \int_{-\pi/R}^{\pi/R} |E_n(e^{j\omega})|^2 \frac{d\omega}{2\pi}, \quad (1)$$

where

$$E_n(z) = \sum_{k=n-M+1}^n e[k]z^{-k} \quad (2)$$

is the windowed z-transform of the output error sequence. The metric in Eqn. 1 measures the in-band noise power of the windowed error sequence by integrating the windowed error spectrum over the frequency region of interest. The metric is time-dependent because the windowed z-transform in Eqn. 2 is time-dependent. The integer constant M represents the memory of the system and limits the number of previous error samples that directly effect the metric. The aspect of finite memory ($M < \infty$) facilitates system realization and is largely ignored in $\Delta\Sigma$ modulators due to the ubiquity of simple integrators. While this finite memory approach was considered by Spang and Schultheiss [5], their development was analytic and required that the number of quantizer levels increased with the amount of memory, M .

The following theorem presents the optimum generation algorithm for the metric in Eqn. 1

Theorem 1. The optimum generation algorithm for the performance metric in Eqn. 1 is:

$$y[n] = Q \left(x[n] - \sum_{k=1}^{M-1} a_k e[n-k] \right), \quad (3)$$

where

$$a_k = \text{sinc}\left(\frac{k}{R}\right) = \frac{\sin\left(\frac{\pi k}{R}\right)}{\frac{\pi k}{R}} \quad (4)$$

and $Q(x)$ represents the legal quantized value closest to x .

Proof: At time instant n , a decision has to be made to make $y[n]$ equal to one of its legal quantized values. The value that minimizes the time-dependent metric in Eqn. 1 will be chosen. Equation 2 can be written as the sum of a term independent of the choice at time instant n and a term dependent on the choice at time instant n :

$$E_n(z) = \text{fit}_n(z) + e[n]z^{-n} \quad (5)$$

Taking the magnitude squared of the above equation and integrating to generate the metric in Eqn. 1 yields:

$$\xi_n = \hat{\xi}_n + \frac{1}{R}(y[n])^2 -$$

$$\frac{2}{R}y[n] \left(x[n] - \sum_{k=1}^{M-1} a_k e[n-k] \right), \quad (6)$$

where $\hat{\xi}_n$ is independent of the choice of $y[n]$ at time instant n and the coefficients a_k are defined in Eqn. 4. Complete the square for the portion of ξ_n that is dependent on $y[n]$. To minimize ξ_n by our choice of $y[n]$, it is only necessary to minimize the portion of ξ_n dependent on $y[n]$. Therefore, it is equivalent to minimize

$$\left(y[n] - \left(x[n] - \sum_{k=1}^{M-1} a_k e[n-k] \right) \right)^2 \quad (7)$$

Therefore, Eqn. 3 is the optimum solution. When the argument in Eqn. 3 is zero, $y[n]$ can be chosen arbitrarily without affecting the minimization of the metric. ■

The synthesis approach system diagram is shown in Figure 1. It is important to note that the generation algorithm is dependent only on the present value of the input signal, $x[n]$. This makes sense since we select the $y[n]$ sequence term by term, suggesting that the future of $x[n]$ is unknown. The output sequence in Eqn. 3 is the quantization of the input signal plus a linear time-invariant filtered version of previous output errors. This bears a strong resemblance to the output of arbitrary $\Delta\Sigma$ modulators [2]. In fact, the following corollary shows that the first-order AX modulator is a special case of this synthesis approach.

Corollary 1. *The first-order $\Delta\Sigma$ modulator is a special case of the synthesis approach where the oversampling ratio, R , the memory, M , and their ratio, it/M ,*

tend to infinity. This corresponds to the case of a nearly-constant input signal.

Proof: As the ratio R/M tends to infinity, each of the coefficients defined in Eqn. 4 tend to unity since $0 \leq k < M$, and the output can be written as:

$$y[n] = Q\left(x[n] - \sum_{k=1}^M e[n-k]\right) \quad (8)$$

In the standard $\Delta\Sigma$ literature, it is more common to express the output error sequence, $e[n]$, in terms of the error introduced by the data conversion. Let $c[n]$ be the quantization error introduced by the quantization operation, Q , in Eqn. 3. Then:

$$\begin{aligned} y[n] &= x[n] - \sum_{k=1}^M c[n-k] + e[n] \\ &= x[n] + e[n] \end{aligned} \quad (9)$$

Therefore as M tends to infinity:

$$\sum_{k=0}^{\infty} c[n-k] = c[n]. \quad (10)$$

It follows that under the asymptotic conditions described above, $e[n] = c[n] - c[n-1]$ which leads to the standard first-order $\Delta\Sigma$ result:

$$y[n] = x[n] + c[n] - c[n-1]. \quad (11)$$

As a check, the transfer function that the conversion error, $c[n]$, sees is $(1 - z^{-1})$, which has a spectral null at DC. ■

3 In-band Noise Power

The in-band noise power is defined to be the amount of error power in the frequency region $|\omega| < \pi/R$. To calculate the in-band noise power, it is assumed that the conversion error, $c[n]$, made by the quantizer is a zero-mean, white, random variable uniformly distributed over one quantization interval. While this assumption is formally not exactly correct, in practice it can be a good first order approximation, and it provides a tractable method of evaluating different architectures.

As in the $\Delta\Sigma$ literature, the difference equation relating the output error, $e[n]$, to the conversion error, $c[n]$, defines the *noise transfer filter*, $N(z)$, that the conversion error passes through to create the output error. Based on Eqn. 9, this difference equation is easily shown to be:

$$e[n] = \sum_{k=0}^{M-1} \text{sinc}\left(\frac{k}{R}\right) c[n-k]. \quad (12)$$

Taking X-transforms we obtain:

$$\phi_e(z) = \frac{\phi_c(z)}{A(z)} = N(z)\phi_c(z), \quad (13)$$

where

$$A(z) = \sum_{k=0}^{M-1} \text{sinc}(k/R)z^{-k}, \quad (14)$$

and $\phi_e(z)$ and $\phi_c(z)$ are the Z -transforms of the output error and the conversion error, respectively.

The transfer function $A(z)$ can be viewed as a rectangularly-windowed FIR approximation to the IIR transfer function whose frequency response is R for $|\omega| < \pi/R$ and zero elsewhere. The approximation improves at frequencies away from $\pm\pi/R$ as the memory, M , increases. Therefore, over the frequency range, $|\omega| < \pi/R$, the magnitude of the noise transfer filter response, $|N(e^{j\omega})|$, approaches $\frac{1}{R}$ as the memory increases. It follows that the in-band noise power is:

$$N_{IB} = \sigma_c^2 \int_{-\pi/R}^{\pi/R} |N(e^{j\omega})|^2 \frac{d\omega}{2\pi} = \frac{\sigma_c^2}{R^3}, \quad (15)$$

where σ_c^2 is the variance of the conversion error.

The dependence of the in-band noise power on the inverse third power of the oversampling ratio is the same here as it is for first-order $\Delta\Sigma$ modulators [2]. However, an important difference is that the above relationship is valid for arbitrary oversampling ratios using this synthesis approach, while the $\Delta\Sigma$ result is valid only for large oversampling ratios. The next theorem elaborates on this result.

Theorem 2. *The synthesis algorithm provides superior in-band noise performance over the first-order $\Delta\Sigma$ modulator for non-constant input signals.*

Proof: When the input signal is constant, the oversampling ratio is infinite, and Eqn. 18 shows that the in-band noise power is zero for the synthesis algorithm. From the proof of Corollary 1, the spectral null at DC in the noise transfer function for the first-order $\Delta\Sigma$ modulator shows that the in-band noise power is also zero.

Using the result of Eqn. 11 and Corollary 1, the noise transfer function for the first-order $\Delta\Sigma$ modulator is $N(e^{j\omega}) = (1 - e^{-j\omega})$. Therefore the in-band noise power for the first-order $\Delta\Sigma$ modulator is:

$$\begin{aligned} N_{IB|\Delta\Sigma 1} &= \sigma_c^2 \int_0^{\pi/R} |1 - e^{-j\omega}|^2 \frac{d\omega}{\pi} \\ &= \frac{2}{R} \sigma_c^2 \left(1 - \text{sinc}\left(\frac{1}{R}\right) \right) \end{aligned} \quad (16)$$

The ratio of the in-band noise powers is a function of R and is found by dividing Eqn. 16 by the result of Eqn. 15:

$$f(R) = \frac{N_{IB|\Delta\Sigma 1}}{N_{IB}} = 2R^2 \left(1 - \text{sinc}\left(\frac{1}{R}\right) \right). \quad (17)$$

The ratio is greater than one if and only if $\text{sinc}(\pi/R) < \frac{\pi}{R} \left(1 - \frac{1}{2R^2} \right)$. Using the product series expansion for

the $\text{sinc}(\theta)$ function, the inequality

$$\begin{aligned} \text{sinc}\left(\frac{\pi}{R}\right) &= \frac{\pi}{R} \prod_{k=1}^{\infty} \left(1 - \frac{1}{R^2 k^2} \right) \\ &= \frac{\pi}{R} \left(1 - \frac{1}{R^2} \right) \prod_{k=2}^{\infty} \left(1 - \frac{1}{R^2 k^2} \right) < \frac{\pi}{R} \left(1 - \frac{1}{2R^2} \right) \end{aligned} \quad (18)$$

is true because, after the obvious cancellation, $\left(1 - \frac{1}{R^2} \right) < \left(1 - \frac{1}{2R^2} \right)$ and $\left| 1 - \frac{1}{R^2 k^2} \right| < 1$ are true for $R > 1$ and $k \geq 2$.

It is natural to wonder how well the synthesis approach compares to conventional $\Delta\Sigma$ modulators of arbitrary order. In-band noise power calculations for an L^{th} order $\Delta\Sigma$ modulator can be made for arbitrary R by replacing the $|1 - e^{-j\omega}|^2$ integrand in Eqn. 16 by $|1 - e^{-j\omega}|^{2L}$. The solution to this integral is easily found. While high-order $\Delta\Sigma$ modulators will have lower in-band noise performance than the synthesis approach for large oversampling ratios they will perform more poorly for lower oversampling ratios. It is of interest to see what the critical oversampling ratio is in order for the L^{th} order $\Delta\Sigma$ modulator to have the same in-band noise performance as the synthesis approach. This can be viewed as the minimum oversampling ratio at which the synthesis approach is outperformed. These values have been computed as a function of the order, L , and are presented in Figure 2. At oversampling ratios below approximately 2.862, no $\Delta\Sigma$ modulator of any order outperforms the synthesis approach. Therefore, this simple analysis suggests the synthesis algorithm can outperform $\Delta\Sigma$ modulators of arbitrary order in high-bandwidth applications requiring a highly linear data converter.

4 Simulations

Figure 3 shows the theoretical in-band noise performance of the synthesis approach and the first-order $\Delta\Sigma$ modulator for oversampling ratios between 1 and 10. These low oversampling ratios are of interest based on the results at the end of the last section. The in-band power is scaled by the variance of the conversion error, σ_c^2 . As proven in Theorem 2, the synthesis approach theoretically performs better than the first-order $\Delta\Sigma$ modulator.

Simulations were performed for oversampling ratios equal to 2, 4 and 8. The input signal for each 128 K-point simulation was a stationary Gaussian process with zero-mean, unit variance and frequency support constrained to the region $|\omega| < \pi/R$ by a tenth-order elliptic filter with 0.3 dB passband ripple and 70 dB stopband attenuation. The same filter was used on the output error sequences before evaluating the experimental in-band noise power. Single-bit converters with output values $\pm c = \pm 2.5$ were used in all simulations. The converter output magnitude, $|c|$, was chosen so that the unit-variance Gaussian would overload, or saturate, the converter infrequently.

The circles (o) and asterisks (*) in Figure 3 are the experimental outcomes for the $\Delta\Sigma$ modulator and the synthesis approach with memory of $M = 100$, respectively. While the experimental results indicate that for these oversampling ratios the synthesis approach outperforms the first-order $\Delta\Sigma$ modulator as predicted by theory, the margin by which it does so and the actual values of the in-band noise power depart, from what is expected. These variations are explained by the fact that for each simulation the spectrum of the conversion error process, $\epsilon[n]$ was slightly colored. While conversion error variance was very close to the predicted values of $\sigma_\epsilon^2 = c^2/3$, the conversion error spectrum was slightly high-pass for the $\Delta\Sigma$ modulator and resulted in lower in-band noise power. Conversely, the conversion error spectrum was slightly low-pass for the synthesis approach resulting in a greater in-band noise power. The theoretical expressions derived in Section 3 assumed white conversion noise, which is valid to a first-order approximation judging from the experimental outcomes in Figure 3 and separate simulations that studied the conversion error specifically.

The simulations also verified predictions about the nature of the noise transfer function for the synthesis approach. Spectra of the output error were obtained and found to have constant response over the passband frequencies. The relationship between in-band noise power and memory was investigated and results are presented in Figure 4 for oversampling ratios of 4 and 8. It was shown in Section 3 that the magnitude of the noise transfer function in the passband tended to $\frac{1}{R}$, where R is the oversampling ratio, as the memory, M became large. Figure 4 suggests that this will be true when the ratio of the memory to the oversampling ratio exceeds unity, or when $M > R$. This has important implications for the complexity of synthesis approach systems. The number of delays and multipliers required in a system that approaches the asymptotic performance described in Section 3 may be small when the oversampling ratio is not large.

5 Conclusions

The synthesis approach is the optimal solution to an oversampled data conversion problem based on the minimization of a metric that measures the amount of noise power residing in a particular frequency region. While the synthesis approach was theoretically and experimentally shown to have better in-band noise performance than the first-order $\Delta\Sigma$ modulator, the synthesis approach has greater complexity. The added complexity may be worthwhile, however, in systems with low oversampling ratios. In addition, experimental results have shown that in such systems the necessary complexity to achieve asymptotic performance is not large.

The limiting aspect of the analytical model presented here was the behavior of the conversion error. This was the primary reason for discrepancy between the theoretical predictions for in-band noise power and experimental observations. Future work needs to address more complicated models for the conversion error spectra as a function of the input signal. This work

is necessary in order to further evaluate the utility of the synthesis approach.

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Figure 1: The Synthesis Approach System Diagram

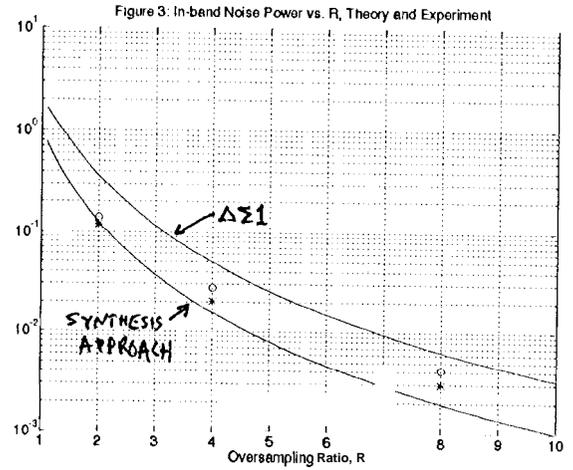
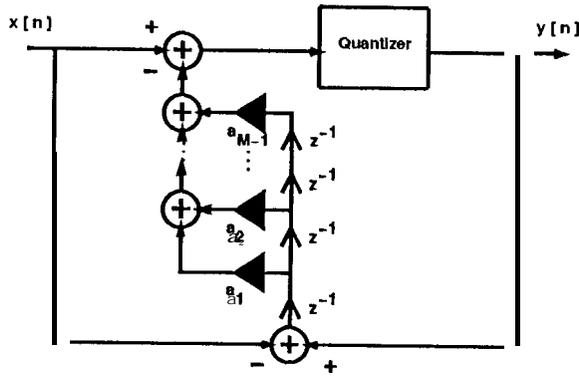


Figure 2: Minimum R to Outperform Synthesis Approach

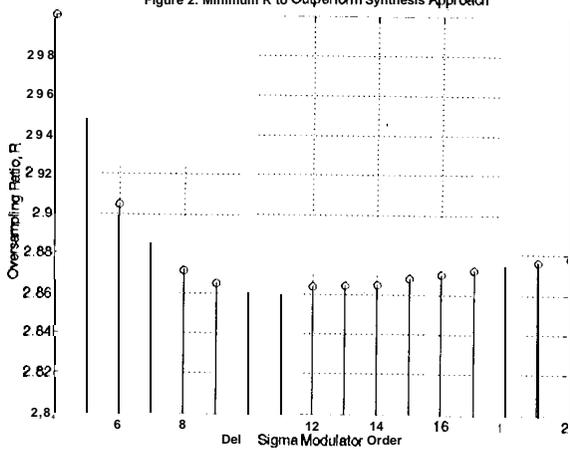


Figure 4 In-band Noise versus Memory, R=4,8

