A COMPLEXITY THEORETIC PERSPECTIVE ON ROBUSTNESS ANALYSIS

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Abstract

Let δ_Σ be a measure of the relative stability of a stable dynamical system Σ defined over the n-dimensional Euclidean space. Let $\tau_{\mathcal{A}(\Sigma)}$ be a measure of the computational efficiency of a particular algorithm A which verifies the stability property of Σ by computing a certificate of stability ***P.*** We demonstrate the existence of a particular measure δ_Σ and an algorithm A such that, $\delta_\Sigma \tau_{\mathcal{A}(\Sigma)} = O(n)$. In addition, we show that δ_Σ determines the size of the certificate ***P.*** These results provide the foundation for an algorithmic theory of stability and robustness.

1 Introduction

1.1

The purpose of the paper is to present the steps that have been taken towards an algorithm theory of stability and robustness.[10,11,12] The algorithmic theory has its roots in the robustness analysis of dynamical systems on one hand, and computational complexity of the optimization problems on the other. 15 The results in the present paper substantiate, an further motivate, our perspective on the deep relationship between these two areaa of research.

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In control and system theory we use computation to study the stability issues of the dynamical systems, but in order to come up with the computational methods, we often use stability results (for example to guarantee reasonable convergence behavior). In effect we have been led to investigate the behavior of one dynamical system by examining the stationary point of the other.

Consider for example the stability analysis of the matrix difference equation,

$$x_{k+1} = A x_k, \quad A \in R^{n\times n}. \tag{1.1}$$

In order to study the stability properties of (1.1), we study the solution of the Lyapunov inequalities,

$$X > o \qquad AXA' - X < 0. \tag{1.2}$$

Let us call the solution to (1.2), if it exists, $\bar{X}$, and refer to it as the ***certificate*** of stability. In order to obtain a feasible point for (1.2), let us employ a variant of the interior point methods (ipms),[7,13] not because ipms are the best methods for solving inequalities of the form (1.2), but due to their general nature which makes them applicable to much larger classes of inequality systems.

Given (1.2) we consider the optimization problem,

$$\inf\ t \tag{1.3}$$
$$AXA' - X < tI, \tag{1.4}$$
$$X > 0. \tag{1.5}$$

It should be clear that for a large enough value t_0 for t, the above optimization problem is at least ***feasible.*** Our algorithm can be constructed based on the following idea: starting from a feasible pair (t_0, Po), at every step reduce the value of t without leaving the feasible region (by appropriate centering, etc.), and stop the algorithm when you reach a negative value for t. This idea parallels that of a variant of the ipms called the barrier method; by carefully choosing the desired reduction in the value oft and an appropriate way of staying away from the boundary of the feasible region, we can in fact prove very nice theoretical complexity bounds for checking the stability of the system (1.1): given that A is stable this method is guaranteed to output the certificate of stability $\bar{X}$ (approximately) in a reasonable time.

We notice that they way we go about constructing an algorithm for solving (1.2) is independent of the actual instance of the problem, that is, its "flowchart" does not depend on a particular matrix A. On the other hand, as long as this flowchart involves any kind of branching (statements of the form: if $x > 0$ then start all over again), the running time of the algorithm depends on the particular instance of the problem- ***the running time of the algorithm depends on the properties of the particular input matrix A.*** Algorithmic Stability Theory is concerned with the computational efficiency of the algorithms which are used to study systems and control problems, expressed in terms of the fundamental quantities in control theory, for example stability margins. What is rather surprising is that this correspondence can be established in a very concrete and elegant framework." For example, we show that not only the robustness measures find their way in the complexity analysis of a particular class of algorithms for solving the LMIs,[4] but also the size of the certificate is a function of these quantities. This phenomena transcends beyond the stability analysis in terms of the Lyapunov inequalities; in fact these parameters appear in the computational efficiency estimates for checking the posit ive real and bounded, realness of transfer matrices, putting the corresponding results at the center stage of the modern robust control theory.

The organization of the paper is as follows. In the next section we initially consider the Lyapunov equation, and demonstrate that the product of the running time of the conjugate gradient method and a particular variant of the interior point methods (ipms) on one hand, and the corresponding robustness measure for linear systems on the other, are inversely proportional. We then to proceed to show the importance that the robustness measures play in defining the solution set of the Lyapunov inequality. The result which extends the above observations to the more general problem of checking the positive realness of a transfer matrix are then presented. The paper is concluded with a brief afterthought on the implications of the results. Some of the proofs are omitted for brevity; the interested reader is referred to the two manuscripts.[10,12]

A few words on the notation before we present our main results. For two symmetric matrices $A = A'$ and $B = B'$, $A > B$ ($A \geq B$) indicates that $A - B$ is positive definite (positive semi-definite, respectively). $\tau_{\mathcal{A}(\Sigma)}$ designates the running time of the algorithm $\mathcal{A}$ which verifies the stability properties of the dynamical system Σ. The notation $f(n) = O(g(n))$ indicates that there exist positive constants c and m such that $0 \leq f(n) \leq cg(n)$ for all $n \geq m$; $\text{herm}\{A\}$ denotes the hermitian part of the matrix A, $\frac{1}{2}(A+A')$. For a given transfer matrix $H(jw)$, $H^*(jw)$ denotes its conjugate transpose.

2 Conjugate Gradients and Stability Analysis

2.1

We now provide some basic facts regarding the numerical algorithm which is at the cornerstone of the main theorem of this section, namely the conjugate gradient method (cgm).

The cgm is devised to solve a given system of linear equation

$$Ax = b, \tag{2.1}$$

where $A \in R^{n\times n}$ is a symmetric positive definite matrix, and $b \in R^n$. The cgm proceeds to solve the linear system (2. 1) for its solution z^*, by considering it as the minimization problem,

$$\min_x \frac{1}{2}x'Ax - b'x$$

Given the pair (A, b) and the initial point x_0, the method produces the sequence $\{x_i\}_{i\geq 1}$, according the following rules:[8]

1. $k = 0; d_0 := -g_0 = b - Ax_0$
2. $\alpha_k = -g_k'd_k/d_k'Ad_k = g_k'g_k/d_k'Ag_k$
3. $x_{k+1} = x_k + \alpha_k d_k$
4. $\beta_k = g_{k+1}'Ad_k/d_k'Ad_k = g_{k+1}'g_{k+1}/g_k'g_k$

5. $d_{k+1} = -g_{k+1} + \beta_k d_k$

6. Go to 2,

where $g_k := Ax_k - b$. The cgm can be viewed as an optimal process, where "optimality" is with respect to the following interesting property: The point x_i generated by the conjugate gradient method satisfies

$$\min_{p\in\mathcal{P}_i} \|p(A)e_0\| = \|x^* - x_i\|_A \qquad (2.2)$$

where $\|y\|_A := (y'Ay)^{1/2}$ for all $y \in R^n$, $\mathcal{P}_i$ is the set of polynomials p of degree at most i, such that $p(0) = 1$, and $e_0 = x^* - x_0$.

A direct consequence of the optimality relation (2.2) is that if A haa $k \leq n$ distinct eigenvalues, then the cgm terminates in k steps, since the minimal polynomial ψ of A (the minimum degree polynomial such that $\psi(A) = O$) would be of degree k. In particular, in view of the Cayley-Hamilton theorem the cgm terminates in at most n steps.[5]

Let us order the eigenvalues of the matrix A as $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Another consequence of the optimality property (2.2) (via the Chebyshev iteration and polynomials) is that,[2,14]

$$\frac{\|x_i - x^*\|_A}{\|x_0 - x^*\|_A} \leq \min_{p\in\mathcal{P}_i} \|p(A)\| \leq \min_{p\in\mathcal{P}_i} \max_{\lambda_i} |p(\lambda_i)|$$

$$\leq \min_{p\in\mathcal{P}_i} \max_{\lambda_1\leq\lambda\leq\lambda_n} |p(\lambda)| = \frac{1}{T_i(\frac{\lambda_n+\lambda_1}{\lambda_n-\lambda_1})} = 2\frac{\sigma^i}{1+\sigma^{2i}} \qquad (2.3)$$

where T_i is the Chebyshev polynomial of the first kind,

$$T_i(z) = 1/2[(z+\sqrt{z^2-1})^i + (z-\sqrt{z^2-1})^i]$$

and

$$\sigma = \frac{1-\sqrt{\lambda_1/\lambda_n}}{1+\sqrt{\lambda_1/\lambda_n}}$$

The second equality from the right in (2.3) follows from an important property of the Chebyshev polynomials.

Define $p := \sqrt{\lambda_n/\lambda_1}$ to be the relative condition number of $\sqrt{A}$. Then it can be shown that,

$$\|x_i - x^*\|_A \leq 2(\frac{1-\rho^{-1}}{1+\rho^{-1}})^i \|x^* - x_0\|_A$$

Thus, in order to guarantee that for $O < \epsilon < 1$, $\|x^* - x_i\|_A/\|x^* - x_0\|_A \leq \epsilon$, one must have

$$(\frac{1-\rho^{-1}}{1+\rho^{-1}})^i \leq \frac{\epsilon}{2}$$

and thereby,

$$i = O(\log_2 \frac{2}{\epsilon}\rho)$$

Hence if $i = \Omega(\log \frac{2}{\epsilon}\rho)$, then $\|x_i - x^*\|_A \leq \epsilon\|x_0 - x^*\|_A$.

For the case that the matrix A is not positive definite, and $A'A$ is not singular, one can consider solving $A'Ax = A'b$, and the above statements still hold with A being replaced by $A'A$.

2.2

We now consider the efficiency of the conjugate gradient method in connection with solving the linear system arising from the Lyapunov equation.

The problem considered by Lyapunov is as follows: Given a matrix $A \in R^{n\times n}$, determine whether $x = O$ is the globally asymptotically stable equilibrium point of Σ_1:

$$\Sigma_1: \quad \dot{x} = Ax. \qquad (2.4)$$

In particular, one is interested to know whether the trajectories of Σ_1 goes back to the origin, if Σ_1 is ***disturbed* by any non-zero initial condition. This** property on the other hand, is equivalent to A being Hurwitz, i.e., all eigenvalues of A have negative real parts.[5]

As it is well known Lyapunov proved that the origin is the globally asymptotically stable equilibrium point of Σ_1 if and only if, given a matrix $Q > 0$, there exists a matrix $P > 0$ such that,[9]

$$A'P + PA = -Q \qquad (2,5)$$

Suppose that the matrix A is in fact Hurwitz, and consider solving (2.5) using Kronecker products, as first proposed by Bellman.[3] Our goal is to demonstrate that the ***running* time of *the conjugate gradient method for approximately* solving *the Lyapunov equation conveys an estimate of the relative stability of A.***

Let us rewrite (2.5) as

$$B \ (vet \ P) = -vet \ Q \qquad (2.6)$$

where

$$B := I \otimes A' + A' \otimes I \in R^{n^2\times n^2}.$$

Each eigenvalue of B *is* the sum of a pair of eigenvalues of A.

In order to use the cgm, we consider solving the system of linear equation

$$B'B(\mathrm{vec}\, P) = -B'(vet\ Q).$$

Since the origin is not in the spectrum of $B'B$ (A is assumed to be Hurwitz), starting with an initial matrix P_0, if

$$i = \Omega(\varrho \log \frac{2}{\epsilon})$$

then the cgm produces the matrix P_i such that,

$$\|P^* - P_i\|_{B'B} \leq \epsilon\|P^* - P_0\|_{B'B}$$

where,

$$\begin{aligned} \varrho &:= \lambda_{\max}(\sqrt{B'B})/\lambda_{\min}(\sqrt{B'B}) \\ &= \sigma_{\max}(B)/\sigma_{\min}(B) = \|B\|/\sigma_{\min}(B) \quad (2.7) \end{aligned}$$

Since $\sigma_{\min}(B) \geq \lambda_1(\mathrm{herm}(-B)) = |\lambda|_{\min}(A + A')$,

$$\varrho \leq \|B\|/\lambda|_{\min}(A + A').$$

Let us define

$$\delta_{\Sigma_1} := |\lambda|_{\min}(A + A')/\|B\|$$

Hence if $i = \Omega(1/\delta_{\Sigma_1})$, then $\|P^* - P_i\|_{B'B} \leq \epsilon\|P^* - P_0\|_{B'B}$. Thereby, in order to obtain an ϵ-approximation of the **certificate** of stability P^* (in the $B'B$ norm), the termination time of the cgm $\tau_{cgm(\Sigma_1)}$, is at least of the order of the magnitude of $1/\delta_{\Sigma_1}$. Denote by $\tau_{cgm(\Sigma_1)}$ the time it takes the cgm to approximately solve the Lyapunov equation associated with Σ_1. One thus has

$$\tau_{cgm(\Sigma_1)}\delta_{\Sigma_1} = o(1).$$

A closer look at the quantity δ_{Σ_1} reveals that it can be viewed as a ***robustness measure*** for the system Σ_1, since,

$$\delta_{\Sigma_1} = |\lambda|_{\min}(A + A')/\|B\| = |\bar{\Delta}|/\|B\|$$

where

$$A := \{\inf A \in R : A + AZ \text{ is not Hurwitz}\}. \quad (2.8)$$

Consequently we have proved the following theorem.

Theorem 2.1 *Given the system Xl, there* is *an algorithm* d *and a stability robustness measure* δ_{Σ_1} *such that,*

$$\delta_{\Sigma_1}\tau_{\mathcal{A}(\Sigma_1)} = O(1).$$

We now state a result which demonstrate the role that robustness measures play in providing a bound for the certificate of stability.

Given a matrix $A \in R^{n\times n}$ such that (1.2) is feasible, define

$$\alpha := \inf_{\Delta} \frac{\|\Delta\|}{\|A\|},$$

such that A + A does not lead to a feasible system (1.2). Let

$$\delta_A := \frac{1}{\log(\sqrt{n} + \frac{1}{\alpha})} \quad (2.9)$$

For obvious reasons, let us call δ_A the robustness parameter of the matrix **A.** One then has the following correspondence between the size of the certificate of stability and robustness parameter δ_A.

Theorem 2.2 *The certificate of stability for a given matrix A, X, satisfies,*

$$\|\bar{X}\| = O(\frac{n}{\delta_A}), \quad (2.10)$$

provided that $\delta_A > 0$.

3 Interior Point Methods and Stability Analysis

3.1

This section is devoted to the results pertaining to the relationship between the computational efficiency of the interior point methods and the stability properties of dynamical systems. We initially state and prove this connection in terms of the Lyapunov inequalities, and then proceed to state the similar results for the positive real systems.

3.1.1 Stability

In order to establish the stability of the origin for the system Σ_1 defined by the matrix $A \in R^{n\times n}$,

$$\Sigma_1: \quad \dot{x} = Ax \quad (3.11)$$

one can check the feasibility of the following system of linear matrix inequalities (L MIs):

$$\mathcal{L}_1: \quad A'P + PA < 0 \tag{3.12}$$
$$P > o \tag{3.13}$$

Let us for the moment forget that these matrix inequalities can somehow be solved via a system of linear equations (Section 2). We approach the problem of finding a feasible point of the set defined by (3.12)-(3.13) via the interior point methods. This will provide us with an opportunity to, rather informally, review the interior point methods **(ipms),** as well as presenting the idea which shall be generalized in the subsequent section. The material on the ipms which follow have been presented in a more general setting.[13,15]

In order to find a feasible point of (3.12)-(3.13), one can consider instead the following optimization problem,

$$\mathcal{L}_2: \quad \inf \ t \tag{3.14}$$
$$\text{St.} \quad A'P + PA < t(A'\bar{P} + \bar{P}A + 1) \tag{3.15}$$
$$P > o \tag{3.16}$$
$$t \geq 0 \tag{3.17}$$

where the matrix P is chosen to be positive definite, e.g., $P = I$. One might wonder why $\bar{P}$ is introduced in (3.15). The reason is that by doing so, a feasible point of $\mathcal{L}_2$ is readily available: $(P_o, t_0) = (\bar{P}, 1)$. Our task is now to initiate the algorithm from $(\bar{P}, 1)$ with the objective value of 1, and try to somehow reduce the objective value to zero (which would be the case if and only if Σ_1 is stable), ***without*** leaving the feasible region of $\mathcal{L}_2$. This is exactly what an interior point method does (more specifically, we have in mind the barrier method).

Few comments, and a reformulation of $\mathcal{L}_2$ precedes our description of the barrier method. Suppose that we were to solve the following optimization problem:

$$\mathcal{L}_3: \quad \inf \ t \tag{3.18}$$
$$\text{s. t.} \quad A'P + PA < tI \tag{3.19}$$
$$P > o \tag{3.20}$$
$$\|P\| < 1 \tag{3.21}$$
$$-1 < t < 2 \tag{3.22}$$

Let $t_{\inf}$ and $t_{\sup}$ denote the value of the infimum and the supremum of the objective functional on the respective region (e.g., $t_{\inf} = O$ in $\mathcal{L}_2$ if Σ_1 is stable). The value oft $_{\inf}$ in $\mathcal{L}_3$ clearly is a measure of relative stability; intuitively, the more negative one can choose t, the more "stable" Σ_1 is, The lower bound for t and the norm constraint on P are chosen for normalization purposes; otherwise the problem would be unbounded, if feasible. The choice of the upper bound for t would be justifies shortly.

It is not clear however how a feasible point for $\mathcal{L}_3$ can be chosen to initiate the algorithm from. We thus consider instead a combination of $\mathcal{L}_2$ and $\mathcal{L}_3$:

$$\mathcal{L}: \quad \inf \ t \tag{3.23}$$
$$\text{s. t.} \quad A'P + PA < t(A'\bar{P} + \bar{P}A + I) \tag{3.24}$$
$$P > o \tag{3.25}$$
$$\|P\| < 1 \tag{3.26}$$
$$-1 < t < 2 \tag{3.27}$$

with P >0 and $\|\bar{P}\| < 1$. The initial point $(\bar{P}, 1)$ is now readily available as an initial point. Again, the value of $t_{\inf}$ for $\mathcal{L}$ somehow conveys information regarding the relative stability of Σ_1, an observation which shall be made more precise shortly.

Let us denote the feasible region of $\mathcal{L}$ by $\mathcal{F}_{\mathcal{L}}$. Note that $\mathcal{F}_{\mathcal{L}} \subseteq SR_+^{n\times n}$ x R and that it is an open and convex set. It turns out that associated with the set $\mathcal{F}_{\mathcal{L}}$, there is a functional b : interior $\mathcal{F}_{\mathcal{L}} \rightarrow R$, which acts as **a** "self-concordant barrier." The term "self concordant" refers to certain properties of the gradient and the Hessian of the functional b evaluated at points in $\mathcal{F}_{\mathcal{L}}$; for the purpose of our presentation, we shall bypass the exact definition and direct the readers' attention to the references given above for the ipm theory.

There are two important points however that need to be emphasized regarding the functional b. First, if $\{x_k\}_{k\geq 1} \in \mathcal{F}_{\mathcal{L}}$ is a sequence that approaches the boundary of $\mathcal{F}_{\mathcal{L}}, b(x_k) \rightarrow \infty$ *as* $k \rightarrow \infty$. Second, there is a parameter K associated with b which determines the computational efficiency of the interior point method for minimizing (or maximizing) a linear functional over $\mathcal{F}_{\mathcal{L}}$, the so-called self-concordant parameter. For brevity, we shall simply write down the self-concordant barrier for $\mathcal{F}_{\mathcal{L}}$ and its associated self-concordant parameter h'. Subsequently, we provide a description of the algorithm for solving $\mathcal{L}$ using b, and its efficiency in terms of the self-concordant parameter of b.

Let $b : FL \rightarrow R$ be defined as:

$$b(P, t) = -12 \log \det P$$

$$-12\log\det(-A'P - PA + t(A'\bar{P} + \bar{P}A + Z)) - 12\log(1 - \|P\|^2) - 12\log(t+1) - 12\log(2-t). \quad (3.28)$$

Note that indeed when $(P_k, t_k) \rightarrow$ boundary of $\mathcal{F}_{\mathcal{L}}$, $b(P_k, t_k) \rightarrow \infty$. The associated self-concordant parameter for the functional b *(3.28)* turns out to be,

$$K := \sqrt{12n + 12n + 48} = O(\sqrt{n}). \quad (3.29)$$

We are now ready to describe the interior point method for solving $\mathcal{L}$. Starting from the initial point $(P_o, t_0) = (\bar{P}, 1)$, and parameter $\mu = \mu_0$, (1) Let $k = 0$, (2) solve the unconstraint optimization problem

$$\min \Phi(P_k, t_k, \mu_k)$$

where $\Phi(P_k, t_k, \mu_k) := \mu_k t + b(P_k, t_k)$, (3) let $\mu_{k+1} = (1 + \frac{1}{6K})\mu_k$, (4) let $k = k + 1$; Go to 2. Of course Step 2 cannot be solved exactly; a lot of research has been devoted to come up with a stopping criterion for this step which is sufficient for proving nice theoretical efficiency of the complete algorithm. One such criterion, rather interestingly, is to take "one" Newton step (with *a* "nice" initial point), and then increase μ_k accordingly (the so-called short step method). Intuitively, as $k \rightarrow \infty$, $\mu_k \rightarrow \infty$, and the sequence of minimal values of $\Phi(P_k, t_k, \mu_k)$ will approach $t_{\inf}$.

The resulting complexity bound stated below is the upshot of the interior point approach.

Theorem 3.1 15 *For solving $\mathcal{L}$, starting with* $(\bar{P}, 1)$ **wing**

$$O(K \log(K + \frac{1}{\epsilon})) \quad (3.30)$$

iterations, ***the above barrier method computes (P*, t*)*** $\in \mathcal{F}_{\mathcal{L}}$, ***and t**** **is** ***known to satisfy***

$$\frac{t^* - t_{\inf}}{t_{\sup} - t_{\inf}} \le \epsilon \quad (3.31)$$

i.e., ***after*** $O(K \log(K + \frac{1}{\epsilon}))$, ***a (relative) c-optimal point*** **is** ***found by the barrier method.***

Theorem 3.1 has few implications which are not difficult to realize.[15]

1. Starting with $(\bar{P}, 1)$, given an $\epsilon > 0$, the barrier method produces the pair (P_i, t_i), known to satisfy

$$(P_i, t_i) \in \mathcal{F}_{\mathcal{L}}, \; t_i - t_{\inf} \le \epsilon, \quad (3.32)$$

if,

$$i = \Omega(K \log(K + \frac{t_{\sup} - t_{\inf}}{\epsilon})) \quad (3.33)$$

2. If $t_{\inf} < \alpha < t_{\sup}$, after

$$O(K \log(K + \frac{t_{\sup} - t_{\inf}}{\min\{t_{\sup} - 0, 0 - t_{\inf}\}})) \quad (3.34)$$

iterations, for which the last pair is (P, t), is guaranteed to satisfy $(P, t) \in \mathcal{F}_{\mathcal{L}}$ and $t = \alpha$.

The second ramification above will now be employed to shed light on the efficiency of the interior point method for determining the stability of Σ_1.

Consider solving $\mathcal{L}$ using the interior point method described above. To check the stability of Σ_1 it is necessary and sufficient to stop the algorithm after the i-th iteration, when $t_i = 0$. According to (2) this is guaranteed after

$$\tau_{\Sigma_1} = O(K \log(K + \frac{t_{\sup} - t_{\inf}}{\min\{t_{\sup}, -t_{\inf}\}})) \quad (3.35)$$

i.e., τ_{Σ_1} is the termination time of the barrier method for checking the stability of Σ_1.

Let $\bar{P} = \frac{1}{2}I$ and start the interior point method described earlier from $(\bar{P}, 1)$. Thus $t_{\sup} \ge 1$ and trivially $t_{\sup} < 2$. Note that $t_{\inf} \le 0$, since if $t_{\inf} > 0$ and the pair $(t_{\inf}, P^*)$ is the solution to $\mathcal{L}$, then for $0 < \epsilon < 1$, $(\epsilon t_{\inf}, \epsilon P^*)$ is also a solution to $\mathcal{L}$, which is a contradiction.

Referring to (3.35), we observe that τ_{Σ_1} is essentially a function of K which is itself a function of n only, and a combination of $t_{\sup}$ and $t_{\inf}$.

Given a Hurwitz matrix A, *let* $\alpha := \inf_\Delta \frac{\|\Delta\|}{\|A\|}$ such that $A + \Delta$ is not Hurwitz. Define

$$\delta_{\Sigma_1} := \frac{1}{\log(\sqrt{n} + \frac{1}{\alpha})} \quad (3.36)$$

Since $-1 < t_{\inf} \le 0$, for small $\epsilon > 0$,

$$A'P + PA < (t_{\inf} - \epsilon)(A'\bar{P} + PA + I) \quad (3,37)$$

$$P > 0 \quad (3.38)$$

$$\|P\| < 1 \quad (3.39)$$

is inconsistent, Thus,

$$\alpha\|A\| \le |t_{\inf} - \epsilon|\|A + I\| \quad (3.40)$$

$$\leq (|t_{\inf}| + \epsilon)(||A|| + 1) \quad (3.41)$$

$$\Rightarrow \frac{\alpha||A||}{||A|| + 1} \leq |t_{\inf}| + \epsilon \quad (3.42)$$

$$\Rightarrow t_{\inf} \leq \frac{-||A||}{||A|| + 1}\alpha \quad (3.43)$$

since $t_{\inf} \leq 0$.

Consequently, since $0 \leq -t_{\inf} \leq t_{\sup} < 2$,

$$\tau_{\Sigma_1} = O(K \log(K + \frac{3}{-t_{\inf}}))$$
$$= O(K \log(K + \frac{1}{\alpha}))$$

Thereby we have established the following theorem.

Theorem 3.2

$$\tau_{\Sigma_1}\delta_{\Sigma_1} = O(K) = O(\sqrt{n}). \quad (3.44)$$

Theorem 3.2 establishes a natural, but very interesting relationship between robustness properties of Σ_1 and the efficiency of ipms for determining whether Σ_1 is stable. More specifically, given that (3.44) holds, fixing n and using the ipm for these lution of the Lyapunov equation, information pertaining to the relative stability of the corresponding system is somehow revealed! This observation has consequences which go far beyond the stability analysis of Σ_1.

3.1.2 Positive Realness

Consider the linear time invariant system Σ_2:

$$\Sigma_2 : \quad \dot{x} = Ax + Bu \quad (3.45)$$
$$y = Cx + Du \quad (3.46)$$

such that the quadruple *(A, B, C, D)* is the minimal realization of the transfer matrix

$$H(s) = C(sI - A)^{-1} B + D$$

in which case we write ***H* ~ *(A, B, C, D)*.** We shall assume that the pairs ***(A, B)*** and ***(A, C)*** are respectively, controllable and observable. The matrix ***A is*** also assumed to be Hurwitz. For further reference let $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$ and $D \in R^{m\times n}$ and without loss of generality assume that $m < n$. Given an initial condition x_0 and a control function η which maps y to u, the equations (3.45)-(3.46) define a trajectory for the feedback system Σ_2.

Given that ***H* ~ *(A, B, C, D)*** and assuming that ***A is*** Hurwitz, the transfer matrix H is called (generalized) strongly positive real (GSPR) if there exists $\epsilon > 0$ such that

$$H(jw) + H^*(jw) > \epsilon I \ \forall w$$

where $H^*(jw)$ denotes the conjugate transpose of the transfer matrix $H(jw)$. The (generalized) Positive Real (GPR) Lemma states that ***H(s) is*** GSPR and stable if and only if the following system of linear matrix inequalities is feasible,[1,16]

$$\mathrm{herm}\{\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\} < 0, \quad (3.47)$$
$$P > 0. \quad (3.48)$$

Let us define two robustness measures for a GSPR system. Denote by $E := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Now let

$$\alpha := \inf_{\Delta} ||\Delta||/||E|| \quad (3.49)$$

such that there ***does not*** exist a matrix ***P*** that satisfies the following set of linear matrix inequalities,

$$\mathrm{herm}\{\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix}(E + \Delta)\} < 0, \quad (3.50)$$
$$P > 0, \quad (3.51)$$

and let

$$\delta_H := 1/\log(\sqrt{n+m} + \frac{1}{\alpha}). \quad (3.52)$$

The quantity $\delta^1_{\Sigma_2}$ is a measure of the relative perturbation that a stable GSPR system Σ_2 can tolerate, and remain stable and GSPR. The perturbation A, can for example be the result of the uncertainty in the modeling of the plant, or due to the finite accuracy of the computer arithmetic (which for example, is used to check the GSPR property of the system).

The ***robustness measure*** for a GPR systems just introduced is related in an interesting way to the ***computational efficiency*** of the barrier method (when applied to solve the system of LMIs resulting from the GPR Lemma).

Theorem 3.3 10 ***Given the system Σ_2, there*** is ***an algorithm A such that for the robustness measures $\delta^1_{\Sigma_2}$ and $\delta^2_{\Sigma_2}$,***

$$\delta_H \tau_{\mathcal{A}(\Sigma_2)} = O(\sqrt{n+m})$$

Similar to the case of the Lyapunov inequality, the robustness parameter effects, in a very precise manner, the size of the certificate of positive realness.

Theorem 3.4 *The certificate of positive realness for a GSPR system, P, satisfies,*

$$\|\bar{P}\| = O(\frac{n+m}{\delta_H}), \tag{3.53}$$

provided that $\delta_H > 0$,

The precise relationship which were presented above between the computational efficiency of certain numerical algorithms on one hand, and the robustness measures for dynamical systems on the other, can in principle be used to give an ***algorithmic definition*** of the relative stability. In particular, one can ***define*** robustness in terms of purely algorithmic consideration to the effect that,

> the degree of the relative stability of a stable system is the inverse of the time it takes to check its stability via the interior point methods.

The objection to this approach would be that the stability properties of a system should in principle be coordinate free, and thus, should not depend on a particular algorithm. Nevertheless, since at the present time, we are far from obtaining ***optimal*** algorithms for solving stability problems (e.g., linear matrix inequalities), a ***machine independent theory of stability is*** far from being realized, Moreover, in order to check the stability of a dynamical system, an algorithm ***has*** to be introduced (on some particular model of computation), and thereby one can argue that, stability properties can be viewed with the running time of that algorithm as our frame of reference. The contribution of the paper is thus to demonstrate that the above approach can indeed be adopted for certain important problems in system analysis.

4 Conclusion

The relationship between the parameters δ_A and δ_H and the complexity estimates of the algorithms which are used in system analysis provide a very natural and intuitive way to understand the interaction between systems and optimization algorithms. At a deeper level however, it is indeed fascinating to see that the connection between efficiency and robustness can be stated in such a concise manner, as the results of this paper indicate.

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