ADAPTIVE H.-CONTROL PROBLEMS AND PARAMETER PROJECTION TECHNIQUES

Wei-Min Lu and Fred Y. Hadaegh *

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, Ca 91109-8099

Abstract

Parameter projection techniques play very impor*tant roles* in *adaptive control. In this paper, two scaled parameter projection techniques with respect to nonsmooth convex parameter sets are developed. The two projection techniques are then used* in *adaptive %-control problems to derive the adap*tive *control laws.*

1 Introduction

New technological development in space engineering and science requires sophisticated control systerns with both high performance and reliability. How to achieve the required performance and reliability against various uncertainties haa been a very challenging issue for control system design in recent years. Adaptive and robust control techniques are considered as useful methods to achieve this goal. Generally speaking, adaptive control is effective in dealing with parametric uncertainty, while the robust control schemes are good at handling dynamic uncertainty. However, oftentimes, control systems have both types of uncertainty; therefore, an integrated treatment in control system design is necessary.

In this paper, we consider adaptive robust control design for an uncertain system with both parametric and dynamic uncertainties, which results in an adaptive system with $\mathcal{H}_\infty$-performance. As the $\mathcal{H}_\infty$-performance has a close relation with dissipativity, a dissipation theoretic approach is used in dealing with the adaptive $\mathcal{H}_\infty$-control problem. Dissipation theoretic technique as a generalization of Lyapunov method [13] for adaptive control yields a simple adaptive control law which can significantly reduce implementation cost. However, the methods generally do not guarantee boundedneaa and convergence of parameters. Though the parameter convergence is not the objective of adaptive control, the boundedness of the estimated parameter should be guaranteed in the adaptive control law. One of the methods to guarantee the boundedness is to use the parameter projection techniques.

In applications, the unknown parameters of the parameterized systems are usually in a bounded set. In this paper, it is assumed that allowable parameter sets are compact and convex, but not necessarily smooth, e.g., a cube in the parameter space. During the adaptation, one of the requirements is that the parameter-adjustment mechanism of an adaptive control system keep the adjusted parameters in the parameter sets so as not to invalidate the solvability conditions. Here, both vector anti direct parameter projection techniques are used to achieve this goal. The vector projection, which was originally introduced as a gradient projection method to generate the feasible directions in constrained optimization [11], is probably the most extensively used projection technique in adaptive parameter estimation and adaptive control [4, 16, 15, 5, 7]. However, in these cases, the projections are considered only for smooth sets. The vector projection is generalized to a more general setting such that the non-smooth parameter sets are allowed. The direct parameter projection is relatively new in adaptive control (another version appeared in [2]). It will be seen that this technique is suitable for the adaptive

“ Associate Fellow AIAA

control problems where integral performance specifications arc involved, in particular adaptive 7&-control problem, The two projections not only play a very important role in adaptive control problems, but also are of intereat in their own right,

This paper is a generalization of the work reported in [9]. Other work related to the adaptive $\mathcal{H}_\infty$-control includes [2, 23, 12, 17]. In this paper, the emphasis is the development of general projection techniques for nonsmooth sets and their applications in adaptive $\mathcal{H}_\infty$-control; the existence of adaptive $\mathcal{H}_\infty$-controllers are characterized in terms of solutions of parameter-dependent Hamilton-Jacobi inequalities, and adaptive controllers are constructed from the solutions and the projection techniques. This paper is thus divided into two parts. In Section 2, the general projection techniques are developed; both direct and direct parameter projections techniques are rigorously treated with respect to compact, convex, but possibly nonsmooth parameter sets. In Section 3, an adaptive $\mathcal{H}_\infty$-control problem is stated; the solutions for the adaptive $\mathcal{H}_\infty$-control problem are derived for the case when the original storage functions are independent of the parameters. Both vector and direct parameter projection techniques are used in the derivation of adaptive control laws.

2 Parameter Projection Techniques

In this section, vector and direct parameter (scaled) projections using the techniques from nonsmooth analysis and viability theory [1, 18]are presented. Both projection techniques will play a very important role in the adaptive $\mathcal{H}_\infty$-control design. The special non-scaled projection techniques were considered in [9].

2.1 Invariance and Contingent Cone

Consider a differential equation:

$$\dot{x} = f(x, t) \tag{1}$$

where $f : R^n \times R^+ \rightarrow \mathbf{R}^n$ is continuous in x and measurable in t. Suppose for all $x_0 \in R^n$, the differential equation has a unique solution starting from $x(0) = x_0$ defined for $t \in \mathbf{R}^+$. A set X c R“ is a invariant set of (l), if for all $x_0 \in$ X, its solution will stay in the set X for all t. It is known that if X is an invariant set, so is its closure $\overline{\mathbf{X}}$.

Given a compact set K c R“, we next examine the invariance for the differential equation (1). The invariance is characterized in terms of contingent cones [1]. The contingent cone to K is well defined as a set-valued map T_K: K $\hookrightarrow$ X:

$$T_{\mathbf{K}}(x) := \{v | \liminf_{h \rightarrow 0^+} \frac{d_{\mathbf{K}}(x + hv)}{h} \leq 0\}. \tag{2}$$

where $d_{\mathbf{K}}(x) := \inf_{z \in \mathbf{K}} \|x - z\|$. For all z $\in$ K, the value $T_K(x)$ is a closed cone. If K is convex, then $T_K(x)$ is a tangent cone. Here, we are concerned with only the convex sets,

Suppose K is convex and O $\in Int(\mathbf{K})$. The Minkowski function of K is defined as (see, e.g., [11])

$$\Psi_{\mathbf{K}}(x) = \inf\{\lambda \in R^+ : x \in \text{ AK}\};$$

Given r $\geq$ O, we define the set $\mathbf{K}_r$ as

$$\mathbf{K}_r := \{x \in \mathbf{R}^n : \Psi_{\mathbf{K}}(x) \leq r\}.$$

Then K = K_1, The Minkowski functions of convex sets are convex, but not differentiable in general [11]. The contingent cone to a convex set is a lower semi-continuous set-valued map on K with closed convex value [1]. The following lemma provides a more explicit representation of contingent cones to convex sets [18].

Lemma 2.1 If K *is convex, then*

$$T_{\mathbf{K}}(x) = \overline{\{y : \exists t > 0 : x + ty \in \mathbf{K}\}}.$$

Another useful notion related to a convex set is its normal cone. If K c $\mathbf{R}^n$ is convex, then we can define its normal cone as follows:

$$N_{\mathbf{K}}(x) = \{z \in \mathbf{R}^n : y^T z \leq 0, \ \ \forall y \in T_{\mathbf{K}}(x)\}$$

Therefore, $N_{\mathbf{K}}(x)$ = {O} if $x \in Int(\mathbf{K})$.

From the above characterizations, we give the following propositions.

Proposition 2.2 ***Consider a linear invertible map*** $Q : \mathbf{R}^n \rightarrow \mathbf{R}^n$. ***Suppose*** K ***is a convex compact set in*** R“, x $\in$ K; ***let*** M = QK ***which is also convex compact, and*** $y = Qx$. ***Then the Minkowski junctions, contingent cones, and normal cones satisfy*** $\Psi_{\mathbf{M}}(y) = \Psi_{\mathbf{K}}(x)$, $T_{\mathbf{M}}(y) = QT_{\mathbf{K}}(x)$, ***and*** $N_{\mathbf{M}}(y) = Q^{-1}N_{\mathbf{K}}(x)$.

Proposition 2.3 *Given a convex compact set* $\mathbf{K} \subset \mathbb{R}^n$, *it is an invariant set of differential equation (1) if and only if for all* $x \in \mathbf{K}$,

$$f(x, t) \in T_{\mathbf{K}}(x) \tag{3}$$

for all $t \in \mathbf{R}+$.

Therefore, the invariance of a set can be characterized by its contingent cone. Given a convex and compact set $K \subset \mathbf{R}^n$, the solutions to differential equation (1) are not necessarily always constrained inside the set K. However, in adaptive control problems, we usually require some parameters, which are governed by differential equations, stay inside given sets during the evolution (see Sections 5 and 6), and some properties still remain. In the following two subsections, we will introduce two projection methods to achieve this goal.

2.2 Direct Parameter Projection

Consider a convex and compact set $K \subset \mathbb{R}^n$. The projection $\Pi^Q_{\mathbf{K}}(x)$ of a point $x \in \mathbf{R}^n$ onto K is defined as follows:

$$\Pi^Q_{\mathbf{K}}(x) = \arg\min_{z \in \mathbf{K}} \|x - z\|_Q := \sqrt{(x-z)^T Q (x-z)} \tag{4}$$

The above projection is well-defined, since $K \subset \mathbf{R}^n$ is convex and compact; in addition, $\Pi^Q_{\mathbf{K}}(x)$ is continuous. We first have the following characterizations.

Proposition 2.4 Given a *convex compact set* $\mathbf{K} \subset \mathbb{R}^n$. *Take* $x \in \mathbb{R}^n$; *the following statements are equivalent:*

(i) $\xi \in K$ *is such that* $\xi = \Pi^Q_{\mathbf{K}}(x)$;

(ii) $(x - \xi)^T Q (x_0 - \xi) \leq 0$.

(iii) $x - \xi \in Q^{-1} N_{\mathbf{K}}(\xi)$.

Proof [(i) $\Leftrightarrow$ (ii)] From the definition,

$$\xi = \Pi^Q_{\mathbf{K}}(x) = \arg\min_{z \in \mathbf{K}} \left\|Q^{1/2}(x - z)\right\|.$$

From Theorem 1 in [11, p.69], the above inequality holds if and only if

$$(Q^{1/2}x - Q^{1/2}\xi)^T (Q^{1/2}x - Q^{1/2}\xi) \leq 0,$$

which is exactly (ii).

[(ii) $\Leftrightarrow$ (iii)] From Lemma 2.1 and the definition of normal cones (see also [18, Proposition 2G]),

$$Q^{-1/2} N_{\mathbf{K}}(\xi) = N_{Q^{1/2}\mathbf{K}}(Q^{1/2}\xi)$$
$$= \{Q^{1/2}v \mid v^T Q (x_0 - \xi) \leq 0, \forall x_0 \in \mathbf{K}\}. \tag{5}$$

From Proposition 2.2,

$$N_{Q^{1/2}\mathbf{K}}(Q^{1/2}\xi) = Q^{-1/2} N_{\mathbf{K}}(\xi),$$

$$Q^{-1} N_K(\xi) = \{v \mid v^T Q (x_0 - \xi) \leq 0, \forall x_0 \in K\}.$$

Therefore, (ii) holds if and only if

$$x - \xi \in Q^{-1} N_K(\xi).$$

In the following, we have the following property of the direct projection.

Proposition 2.5 *Let a convex compact set* $\mathbf{K} \subset \mathbb{R}^n$. *Then for any absolutely continuous function* $x: \mathbf{R} \to \mathbf{R}^n$, *its projection:*

$$\xi(t) := \Pi^Q_{\mathbf{K}}(x(t))$$

is also absolutely continuous.

Proof As Q is invertible, it is sufficient to show that the map $Q^{1/2}\Pi^Q_{\mathbf{K}} : \mathbf{R}^n \to \mathbf{R}^n$ is Lipschitz. In fact, we will show that for all $x, y \in \mathbb{R}^n$,

$$\left\|Q^{1/2}\Pi^Q_{\mathbf{K}}(x) - Q^{1/2}\Pi^Q_{\mathbf{K}}(y)\right\| \leq \left\|Q^{1/2}\right\| \|x - y\|. \tag{6}$$

Indeed,

$$\left\|Q^{1/2}(x - y)\right\|^2$$
$$= \left\|\Pi^Q_{\mathbf{K}}(x) - \Pi^Q_{\mathbf{K}}(y) + (x - \Pi^Q_{\mathbf{K}}(x)) - (y - \Pi^Q_{\mathbf{K}}(y))\right\|^2_Q$$
$$= \left\|\Pi^Q_{\mathbf{K}}(x) - \Pi^Q_{\mathbf{K}}(y)\right\|^2_Q + \left\|(x - \Pi^Q_{\mathbf{K}}(x)) - (y - \Pi^Q_{\mathbf{K}}(y))\right\|^2_Q +$$
$$-2(x - \Pi^Q_{\mathbf{K}}(x))^T Q (\Pi^Q_{\mathbf{K}}(y) - \Pi^Q_{\mathbf{K}}(x)) -$$
$$2(y - \Pi^Q_{\mathbf{K}}(y))^T Q (\Pi^Q_{\mathbf{K}}(x) - \Pi^Q_{\mathbf{K}}(y))$$
$$\geq \left\|\Pi^Q_{\mathbf{K}}(x) - \Pi^Q_{\mathbf{K}}(y)\right\|^2_Q$$
$$= \left\|Q^{1/2}\Pi^Q_{\mathbf{K}}(x) - Q^{1/2}\Pi^Q_{\mathbf{K}}(y)\right\|^2$$

where the inequality follows from the above proposition (ii), e.g.,

$$(x - \Pi^Q_{\mathbf{K}}(x))^T Q(\Pi^Q_{\mathbf{K}}(y) - \Pi^Q_{\mathbf{K}}(x)) \le 0.$$

The direct parameter projection hss the following property which is useful in the adaptive $\mathcal{H}_\infty$**-control** problem.

Theorem 2.6 Given the *convex and compact* K. *Then for any absolute continuous junction z :* $\mathbf{R}^+ \to \mathbf{R}^n$ *with* $x(0) \in$ K, *and the projection.' f(t)=* $\Pi^Q_{\mathbf{K}}(x(t))$*, the following inequality holds:*

$$\int_0^T (\xi(t) - x^*)^T Q(\dot{\xi}(t) - \dot{x}(t))dt \le 0 \quad (7)$$

for all z" $\in$ K *and* $T \ge 0$.

Proof Notice that the left-hand side of (7) is well defined from the previous proposition. **Thus,** given $x^* \in \mathbf{K}$ **and** $T \ge 0$**, we have**

$$\begin{aligned} &\int_0^T (\xi(t) - x^*)^T Q(\dot{\xi}(t) - \dot{x}(t))dt \\ &= (\xi(t) - x(t))^T Q(\xi(t) - x^*)\,|_0^T + \\ &\quad - \int_0^T (\xi(t) - x(t))^T Q d\xi(t) \\ &= (\xi(T) - x(T))^T Q(\xi(T) - x^*) + \\ &\quad - \int_0^T (\xi(t) - x(t))^T Q\dot{\xi}(t)dt \end{aligned}$$

where the last equality holds as $\xi(0)$ = z(O) $\in$ K.

As $\xi(t) = \Pi^Q(x(t))$ for $t \in$ *[0, T],* Proposition 2.4 (iii) gives

$$x(t) - \xi(t) \in Q^{-1}N_{\mathbf{K}}(\xi(t)) \quad (8)$$

for all $t \in [T_0, T]$.

Also from Proposition 2.4 (ii), one has the following:

$$\begin{aligned} &(\xi(T) - x(T))^T Q(\xi(T) - x^*) \\ &= (x(T) - \xi(T))^T Q(x^* - \xi(T)) \le O. \end{aligned}$$

On the other hand, if $\xi(t) \in$ K is differentiable at $t \in$ (7", *T),* there exists a positive sequence $\{h_n\}$ with $h_n \to 0$ *as* $n \to \infty$ such that $\xi(t+h_n) \in$ K; denote

$$d_n(t) := \frac{\xi(t + h_n) - \xi(t)}{h_n},$$

then $d_n(t) \to \dot{\xi}(t)$ *as* $n \to \infty$. *Since* $\xi(t) + h_n d_n(t) = \xi(t + h_n) \in \mathbf{K}, d_n(t) \in T_{\mathbf{K}}(\xi(t))$ by Lemma 2.1. Therefore, $\xi(t) = \lim_{n\to\infty} d_n(t) \in T_{\mathbf{K}}(\xi(t))$ as $T_K(\xi(t))$ is a closed cone. Thus, (8) as ***well* as** the definition of normal cone implies

$$(\xi(t) - x(t))^T Q\dot{\xi}(t) = -(x(t) - \xi(t))^T Q\dot{\xi}(t) \ge 0.$$

for all $t \in$ *[To, T].* Therefore,

$$\begin{aligned} &\int_0^T (\xi(t) - x^*)^T Q(\dot{\xi}(t) - \dot{x}(t))dt \\ &= (\xi(T) - x(T))^T Q(\xi(T) - x^*) - \int_0^T (\xi(t) - x(t))^T Q\dot{\xi}(t)dt \le 0 \end{aligned}$$

In the above proof, we have also proved the following useful result.

Corollary 2.7 *Let* $\xi : \mathbf{R}^+ \to \mathbf{R}^n$ *be an absolutely continuous function. Give a compact and convex set* K $\in$ R"; *if* $\xi(t) \in$ K for *ant* $\in$ R", then

$$\dot{\xi}(t) \in T_{\mathbf{K}}(\xi(t))$$

for almost all $t \in$ R".

2.3 Vector Projection

Vector projection technique is introduced for constrained optimization (see, e.g., [11]). It is an indirect parameter projection technique. It has been widely used in the adaptive parameter estimation and adaptive control (see e.g., [4, 16, 5]).

Consider a convex and compact set K *c* R" and its Minkowski function $\Psi_{\mathbf{K}}$. Given a positive definite scaling matrix Q, we first define the scaled projection of a vector v $\in$ R^n at a point $x \in$ R" on the contingent cone $T_{\mathbf{K}}(x)$ *as* follows:

$$\pi^Q_{\mathbf{K}}(x, v) = \begin{cases} v & \text{if } x \in Int(\mathbf{K}) \text{ or } v \in T_{\mathbf{K}_r}(x); \\ 0 & \text{if } v \in Q^{-1}N_{\mathbf{K}_r}(x); \\ V^T Q w_o w_o & \text{otherwise.} \end{cases} \quad (9)$$

where $r = \Psi_{\mathbf{K}}(x)$ and

$$W. = \arg\max\{v^T Q w | w \in T_{\mathbf{K}_r}(x), \|w\|_Q = 1\}.$$

We have the following theorem.

Theorem 2.8 Given *a convex and compact set* K *C* R^n, *the projection is defined by (9), then we have the following assertions:*

(i) $\pi_{\mathbf{K}}^{Q}(x,v) \in T_{\mathbf{K}}(x)$ *for all* $x \in$ K *and* $v \in \mathbf{R}^n$.

(ii) $\|\pi_{\mathbf{K}}^{Q}(x, v)\|_Q \leq \|v\|_Q$ *for all* $x \in$ K *and* $v \in \mathbf{R}^n$.

(iii) $(z - x^*)^T Q(\pi_{\mathbf{K}}^{Q}(x, v) - v) \leq 0$ *for* $x^* \in$ K, $x \in \mathbf{K}$, *and* $v \in \mathbf{R}^n$.

(iv) *Consider the system (1) under the above projection:*

$$x = \pi_{\mathbf{K}}^{Q}(x, f(x, t)); \qquad (10)$$

then the set K *is an* **invariant set** *for the projected system.*

Proof Given $z \in$ K and $y \in \mathbf{R}^n$.

(i) If $x \in Int(\mathbf{K})$, then $T_{\mathbf{K}}(x) = \mathbf{R}^n$, so $\pi_{\mathbf{K}}^{Q}(x,v) = v \in T_{\mathbf{K}}(x)$; if $v \in T_{\mathbf{K}}(x)$, then $\pi_{\mathbf{K}}^{Q}(x, v) = v \in T_{\mathbf{K}}(x)$; if $v \in Q^{-1}N_{\mathbf{K}_r}(x) = \{w : v^T Q w \leq 0, \forall v \in T_{\mathbf{K}_r}(x)\}$, then $\pi_{\mathbf{K}}^{Q}(x,v) = 0 \in T_{\mathbf{K}}(x)$. We only need to show the last case with $\Psi(x) \geq 1$. In fact, as $w_0 \in T_{\mathbf{K}_r}(x)$ and $v^T Q w_0 \geq 0$, then $\pi_{\mathbf{K}}^{Q}(x, u) = (v^T Q w_0) w_0 \in T_{\mathbf{K}_r}(x)$.

(ii) As the inequality is satisfied trivially in the first two cases, we only need to show the **case** when $\pi_{\mathbf{K}}^{Q}(z, v) = v^T Q w_0 w_0$. Indeed,

$$\left\|\pi_{\mathbf{K}}^{Q}(x,v)\right\|_Q = \|v^T Q w_0 w_0\|_Q$$

$$= \|v^T Q w_0\| \leq \|v\|_Q \|w_0\|_Q = \|v\|_Q .$$

(iii) If $x \in Int(\mathbf{K})$ or $v \in T_{\mathbf{K}}(x)$, then $\pi_{\mathbf{K}}^{Q}(x, v) = v$, *so* $(x - x^*)^T Q(\pi_{\mathbf{K}}^{Q}(x, v) - v) = 0$. *Now we* consider the other cases.

Notice that, for **all** $z^* \in$ K, **as** $z + (z^* - x) = x \in$ K, then $z^* - x \in T_{\mathbf{K}}(x)$ by Lemma 2.1. So if $v \in Q^{-1}N_{\mathbf{K}}(x)$, then $Qv \in N_{\mathbf{K}}(x)$, and by the definition of normal cone,

$$(x - x^*)^T Q(\pi_{\mathbf{K}}^{Q}(x, v) - v) = (x^* - x)^T Q v \leq 0.$$

Next, let's consider the remaining **case**: $\pi_{\mathbf{K}}^{Q}(x, v) = v^T Q w_0 w_0$. We first show the following:

$$Q(v - \pi_{\mathbf{K}}^{Q}(x, v)) \in N_{\mathbf{K}}(x).$$

From the **definition** of the projection,

$$w_0 = \arg\max\{(Q^{1/2}v)^T(Q^{1/2}w) |$$

$$Q^{1/2}w \in Q^{1/2}T_{\mathbf{K}_r}(x), \|Q^{1/2}w\| = 1 \}$$

Therefore,

$$v^T Q w_0 Q^{1/2} w_0 = (Q^{1/2}v)^T (Q^{1/2} w_0)(Q^{1/2} w_0)$$

$$= \arg \min_{z \in Q^{1/2} T_{\mathbf{K}}(x)} \|Q^{1/2} v - z\|,$$

or

$$v^T Q w_0 w_0 = \arg \min_{Q^{1/2} z_0 \in Q^{1/2} T_{\mathbf{K}}(x)} \|v - z_0\|_Q .$$

By the use of Proposition 2.4, we have

$$(v - v^T Q w_0 w_0)^T Q(z - v^T Q w_0 w_0) \leq 0 \quad (11)$$

for **all** $z \in T_{\mathbf{K}}(x)$. **On** the other hand, as $T_{\mathbf{K}}(x)$ is a convex **cone, so for all** $u \in T_{\mathbf{K}}(x)$, $u + v^T Q w_0 w_0 \in T_{\mathbf{K}}(x)$; **so (11) implies**

$$(v - v^T Q w_0 w_0)^T Q u$$

$$= (v - v^T Q w_0 w_0)^T Q(u + v^T Q w_0 w_0 - v^T Q w_0 w_0) \leq 0.$$

Therefore, $Q(u - v^T Q w_0 w_0) \in N_{\mathbf{K}}(x)$ as claimed. Again by the definition of normal cone,

$$(x - x^*)^T Q(\pi_{\mathbf{K}}^{Q}(x, v) - y) = (x^* - x)^T Q(y - \pi_{\mathbf{K}}^{Q}(x, v)) \leq 0.$$

(iv) As K is convex, then its Minkowski function $\Psi_{\mathbf{K}}$ **is** convex, so it **is** absolutely **continuous.** Now for any absolutely continuous function $x(t)$ that is a solution of (10) with $x(0) \in$ K, then $\eta(t) := \Psi_{\mathbf{K}}(x(t))$ is **also** absolutely continuous. It **is** sufficient to show $x(t) \in$ K. Indeed, if it is not true, then there exists $T > 0$, such that $z(T) \in \mathbf{R}^n \backslash \mathbf{K}$; so $\Psi_{\mathbf{K}}(x(T)) > 0$. **suppose** $T_0 < T$ *is* such that

$$T_0 = \inf\{t \geq 0 : z(t) \in \mathbf{R}^n \backslash \mathbf{K}\}$$

Therefore, $x(T_0) \in$ K as K is compact, so $\Psi_{\mathbf{K}}(x(0)) \leq 1$. Let $t \in (T_0, T)$ be a point on $\mathbf{R}^+$ where both $\dot{x}(t)$ and $\frac{\partial \Psi_{\mathbf{K}}}{\partial x}(z(t))$ exist, then there exists $\epsilon(h)$ with $\lim_{h \to 0} \frac{\epsilon(h)}{h} = 0$ such that

$$x(t + h) = z(t) + h\dot{x}(t) + \epsilon(h).$$

Then

$$\dot{\eta}(t) = \lim_{h \to 0^+} \frac{\Psi_{\mathbf{K}}(x(t) + h\dot{x}(t) + \epsilon(h)) - \Psi_{\mathbf{K}}(x(t))}{h}$$

$$= \frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t))\dot{x}(t) = \frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t))\pi_{\mathbf{K}}^{Q}(x, f(x, t))$$

Notice that $\frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t)) \in N_{\mathbf{K}_r}(x(t))$ where $r = \Psi_{\mathbf{K}}(x(t))$; **by** the same argument as (i), we can show $\pi_{\mathbf{K}}^{Q}(x, f(x, t)) \in T_{\mathbf{K}_r}(x(t))$. Thus,

$$\dot{\eta}(t) = \frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t))\pi_{\mathbf{K}}^{Q}(x, f(x, t)) \leq 0.$$

almost everywhere on $[T_0, T]$. Thus,

$$0 < \Psi_{\mathbf{K}}(x(T)) - \Psi_{\mathbf{K}}(x(T_0)) = \eta(T) - \eta(T_0)$$
$$= \int_{\theta}^{T} \dot{\eta}(t)dt \leq 0,$$

which is a cent radict ion. Therefore, K is an invariant set.

Notice that in the above theorem, (iv) is not the conclusion of (i), as we don't assume the solutions of (10) are unique for z(O) $\in$ K (see Proposition 2.3). In the proof of (iv), the projection property of a vector outside K is used which is also true if the projection (2.8) of a vector is defined onto the exterior contingent cones instead of $T_{\mathbf{K}_r}(x)$ with r > 1 (see [1, Definition 5.1. l]). (iii) is a useful property for the adaptive control design.

It is also noticed that the right-hand side of (10) is not necessarily continuous, even if f is continuous. In [16], with some relaxation, the authors define a projection which is Lipachitzian and guaranteea the projected system to have an invariant set larger than the parameter set.

3 Adaptive $\mathcal{H}_\infty$-Control

3.1 Adaptive $\mathcal{H}_\infty$-Control Problem and Dissipativity

In this section, we will consider the adaptive attenuation of disturbances for nonlinear systems with emphasis on the application of projection techniques. The uncertain nonlinear system G(O) to be considered is governed by the following parameterized dynamical equation:

$$\begin{cases} \dot{x} &= f(x,\theta) + g_1(x,\theta)w + g_2(x,\theta)u \\ z &= h(x) + k_1(x)w + k_2(x)u \\ Y &= h_2(x) + k_{21}(x)w + k_{22}(x)u \end{cases} \tag{12}$$

where θ is a r-dimensional vector of unknown constant parameters with $\theta \in \Theta \in \mathbf{R}^r$. In the adaptive cent rol problem, we will consider the parameter-dependence in the following fashion:

$$f(x,\theta) = f_0(x) + \sum_{i=1}^{r} \theta_i f_i(x)$$

$$g_j(x,\theta) = g_{j0}(x) + \sum_{i=1}^{r} \theta_i g_{ji}(x), \; j = 1, 2.$$

It is assumed that $f_i, g_{ji}, k_j \in \mathbf{C}^0$, $f_i(0) =$ O, $h(0) = O$, and $R(x) I - k_1^T(x)k_1(x) > 0$ for all x $\in$ R"; x, w, u, z, and y are state, exogenous disturbance, control input, regulated output, and measured output with dimensions n, p_o, p_2, q, and $n + p_1$, respectively.

The objective of the adaptive $\mathcal{H}_\infty$-controller design is to attenuate the impact z of the exogenous disturbance w and the error induced by the initial guess of the parameter. The magnitudes of signals z, w and z are measured by their $\mathcal{L}_2$-norms. The adaptive controllers to be sought have the following form.

$$K: \begin{cases} \dot{p} = \phi(p, y, u) \\ u = \kappa(p)y \end{cases} \tag{13}$$

where $p \in \mathrm{R}^r$ is the estimation of the real parameter $\theta, \phi \in$ Co, and $\dot{p} = \phi(p, y, u)$ is the parameter update law. For fixed p, $u = \kappa(p)y$ is a 1/0 map from y to u; it is taken as a (possibly modified) gain-scheduled controller in the sequel. An adaptive?& control system is illustrated in Figure 1. The preciae statement of the adaptive control problem is given next.

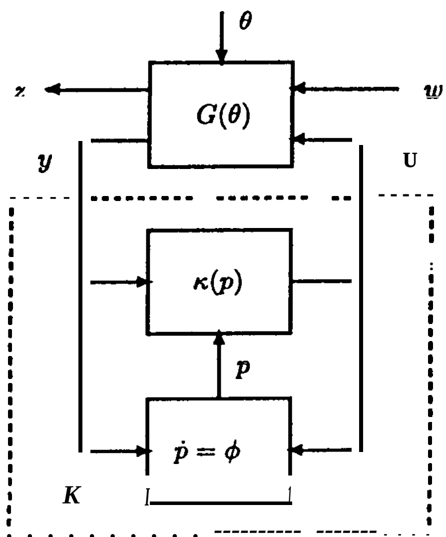

Figure 1: Adaptive $\mathcal{H}_\infty$-Control System

Definition 3.1 (Adaptive $\mathcal{H}_\infty$-**Control Problem)** *Suppose $\epsilon > 0$ is given. The adaptive $\mathcal{H}_\infty$-control design is to seek a controller (13) such that the resulting closed loop system with $x(0) = O$ satisfies*

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt + (p(0) - \theta)^T Q(p(0) - \theta) \tag{14}$$

for all $T \in$ R+, w $\in \mathcal{L}_2[0,\infty)$, $p(0) \in \Theta$, *and* $O \in \Theta$.

The performance can be interpreted as the attenuation of exogenous disturbance and the error of the initial parameter guess. Note that in this statement the initial state is assumed to be at the origin. If the initial state is unknown, the performance can be modified accordingly.

In the following, we consider full-information feedback, in which csse both x and w are available to the control input u. Moreover, we make the following assumptions to simplify the process.

Assumption *3.2 Consider the system (12).*

[A1] The parameter set Θ *w convex and compact, and* $O \in Int(\Theta)$.

[A2] $y = \begin{bmatrix} x \\ w \end{bmatrix}$.

[A3] $k_1(x) = O$ *and* $k_2^T(x)\ [\ h(x)\ k_2(x)\ ] = [\ 0 \quad I\ ]$ *for all* $x \in \mathbf{R}^n$.

Assumption [A2] just restates the full information problem. [A3] is a standard assumption in the?& control problem [3, 8] in most of the derivationa in the following.

Remark 3.3 ***It will be seen that for the parameter-dependent system (12) with above assumptions, if the multiplier*** $9_1(z, O)$ ***of the disturbance w w independent of*** θ***, then only the state information, in stead of full information, is needed to construct the adaptive ?&-control law.***

It is known that if the parameter θ is known, then the $\mathcal{H}_\infty$-control problem has a state-feedback solution if there exists a non-negative function $V: \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$ which is positive definite with respect to x such that the following parameter dependent Hamilton-Jacobi inequality is satisfied for all $\theta \in \Theta$:

$$\frac{\partial V}{\partial x}(x,\theta)f(x,\theta) + \frac{1}{4}\frac{\partial V}{\partial x}(x,\theta)(g_1(x,\theta)g_1^T(x,\theta) + -g_2(x,\theta)g_2^T(x,\theta))\frac{\partial V^T}{\partial x}(x,\theta) + h^T(x)h(x) \leq 0 \tag{15}$$

and the parameter-dependent $\mathcal{H}_\infty$-controller is

$$u = -\frac{1}{2}g_2(x,\theta)\frac{\partial V^T}{\partial x}(x,\theta). \tag{16}$$

Moreover, the closed loop system with the above controller is dissipative with respect to the supply rate $\|w\|^2 - \|z\|^2$, and the function V is a storage function for the closed-loop system satisfying the dissipation inequality:

$$\dot{V}(x,\theta) \leq \|w\|^2 - \|z\|^2 . \tag{17}$$

However, if the parameter is unknown before the system is in operation, we need to design an adap tive mechanism to estimate the parameter on-line and use the estimated parameter to adjust the necessary control action; in which case, the controller

$$u = \psi(x,w,p)$$

is used instead, where ψ is the state feedback (or its modification) defined in (16), p *is* an estimation of 8, and its update law has the following general form:

$$\dot{p} = \phi(p, x, w, u)$$

To guarantee the $\mathcal{H}_\infty$-performance (14) for the closed system, we need to show that the adaptive (closed-loop) system is dissipative with respect to the supply rate $\|w\|^2 - \|z\|^2$; it is enough to find a storage function $W_\theta : \mathbf{R}^n \times \mathbf{R}^r \rightarrow \mathbf{R}^+$ for each $\theta \in \Theta$, such that the following dissipation inequality is satisfied:

$$W_\theta(x(T),p(T)) - W_\theta(x(0),p(0)) \leq \int_0^T (\|w(t)\|^2 - \|z(t)\|^2)dt.$$

Its differential version is satisfied if W is differentiable:

$$\frac{\partial W_\theta}{\partial x}(x,p)(f(x,\theta) + g_1(x,\theta)w + g_2(x,\theta)\psi(x,w,p)) + \frac{\partial W_\theta}{\partial p}(x,p)\phi(p,x,w,\psi(x,w,p)) \leq \|w\|^2 - 11\%11^2. \tag{18}$$

for each $\theta \in \Theta$ and $(z, P) \in$ R' x R^r. Next, we will explicitly construct storage functions such that (18) is satisfied. From the discussion in Section 2.2, suppose the Hamilton-Jacobi inequality (3. 1) has a solution $V(x, 0)$, a possible choice for the storage function is $V(x,p)$ where the unknown parameter θ is replaced by its estimation p. However, it does not reflect the parameter estimation nature for the up date law. On the other hand, the parameter enters the system in an affine fashion. Therefore, a meaningful choice of the storage function of the adap tive control system is the onc with an additional quadratic p-term:

$$W_\theta(x,p) = V(x,p) + (p - \theta)^T Q(p - 0) \tag{19}$$

Note that this idea was first introduced to construct Lyapunov functions for stable adaptive systems [13], and haa been used in many adaptive control problems [6, 16, 23, 7]. For the sake of simplicity, we will assume that the function $V: \mathbf{R}^n \times \Theta \to \mathbf{R}^+$ satisfying the the above Hamilton-Jacobi inequality be continuously differentiable with respect to both arguments. Detailed solutions to the adaptive ?&-control problem with full information feedback in different cases when V is independent of the parameter is presented next.

3.2 Solutions to Adaptive $\mathcal{H}_\infty$-Control Problem

In this section, we mainly consider the case where there exists a positive definite function $V: \mathbf{R}^n \to \mathbf{R}^+$ which is independent of θ such that it satisfies the Hamilton-Jacobi inequality (3.1), i.e.,

$$\frac{\partial V}{\partial x}(x)f(x,\theta) + \frac{1}{4}\frac{\partial V}{\partial x}(x)(g_1(x,\theta)g_1^T(x,\theta)+$$

$$-g_2(x,\theta)g_2^T(x,\theta))\frac{\partial V^T}{\partial x}(x) + h^T(x)h(x) \leq 0. \quad (20)$$

for all $\theta \in$ e.

Let $W: \mathbf{R}^n \times \Theta \to \mathbf{R}^+$ be a positive definite function defined as

$$W(x,p) = \mathrm{V(z)} + (p - \theta)^T Q(p - 8). \qquad (21)$$

where Q is the positive definite matrix defined in the definition. Take W as a storage function candidate of the adaptive $\mathcal{H}_\infty$-control system. Then

$$\dot{W}(x,p) = \dot{V}(x) + 2(p-\theta)^T Q\dot{p}$$

$$= \frac{\partial V}{\partial x}(x)(f(x,\theta)+g_1(x,\theta)w+g_2(x,\theta)u)+2(p-\theta)^T Q\dot{p}$$

$$\frac{\partial V}{\partial x}(x)(f(x,p) + g_1(x,p)w + g_2(x,p)u)+$$

$$+\sum_{i=1}^{r}\{\frac{\partial V}{\partial x}(x)(\theta_i - p_i)(f_i(x) + g_{1i}(x)w + g_{2i}(x)u)\}$$

$$+2(p-\theta)^T Q\dot{p}$$

Notice that if p $\in$ 0, then from the assumption (3.2), then

$$\frac{\partial V}{\partial x}(x)\, f(x,p) \leq -(\frac{1}{4}\frac{\partial V}{\partial x}(x)(g_1(x,p)g_1^T(x,p)+$$

$$-g_2(x,p)g_2^T(x, \mathrm{P})) \frac{\partial V^T}{\partial x}(x) + h^T(x)h(x))$$

Replace the above inequality and use the completion of square, one has

$$\dot{W}(x,p) \leq \|w(t)\|^2 - \|z(t)\|^2 +$$

$$+ \left\|u(t) + \frac{1}{2}g_2^T(X,p)\frac{\partial V^T}{\partial x}(x))\right\|2 .$$

$$w(t) - \frac{1}{2}g_1^T(x,p)\frac{\partial V^T}{\partial x}(x) \quad {}^2 +2(p-\theta)^T Q(\dot{p}-\Phi(x,w,u)), \quad (22)$$

where Φ : R* x $\mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \to \mathbf{R}^r$ is defined as

$$\Phi(x,w,u) = \frac{1}{2}Q^{-1}\begin{bmatrix} \frac{\partial V}{\partial x}(x)(f_1(x) + g_{11}(x)w + g_{21}(x)u) \\ \frac{\partial V}{\partial x}(x)(f_2(x) + g_{12}(x)w + g_{22}(x)u) \\ \vdots \\ \frac{\partial V}{\partial x}(x)(f_r(x) + g_{1r}(x)w + g_{2r}(x)u) \end{bmatrix} \quad (23)$$

From (22), one has that if $p \in \Theta$ and $u = -\frac{1}{2}g_2^T(x,p)\frac{\partial V^T}{\partial x}(x)$, then

$$\dot{W}(x,p) \leq \|w(t)\|^2 - \|z(t)\|^2 + 2(p-\theta)^T Q(\dot{p}-\Phi(x,W,U)), \quad (24)$$

Now integrate both sides of (24) from O to T and notice $W(x(T),p(T)) \geq O$ and $x(0)$ = O, we have

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt + (p(0)-\theta)^T Q(p(0)-\theta)+$$

$$+2\int_0^T (p(t)-\theta)^T Q(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt$$

$$\leq \int_0^T \|w(t)\|^2 dt + (p(0) - \theta)^T Q(p(0) - \theta)$$

$$+2\int_0^T (p(t)-\theta)^T Q(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt. \quad (25)$$

Therefore, if we can find a parameter update law for p such that

$$\int_0^T (p(t) - \theta)^T Q(\dot{p}(t) - \Phi(x(t),w(t),u(t)))dt \leq 0$$

and

$$p(t) \in 0, \forall t \in \mathrm{R}+,$$

then the adaptive $\mathcal{H}_\infty$-control problem is solved. Fortunately, we can use the projection techniques developed in Section 3 to achieve the above requirements.

Theorem 3.4 (Adaptive $\mathcal{H}_\infty$-**Control** with Vector Projection) ***Consider the parameter-dependent system (12). Suppose there exists a non-negative function $V: \mathbf{R}^n \to \mathbf{R}^+$ such that for each***

$\theta \in \Theta$, *(3.2)* is *satisfied. Then given* a *positive definite matrix* $Q \in \mathbb{R}^{r\times r}$, *the adaptive* $\mathcal{H}_\infty$*-control problem has a solution. And an adaptive control law is* given *by*

$$\begin{cases} \dot{p} = \pi^Q_\Theta(p, \Phi(x, w, -\frac{1}{4} g_2^T(x,p)\frac{\partial V^T}{\partial x}(x)) \\ u = -\frac{1}{2} g_2^T(x,p)\frac{\partial V^T}{\partial x}(x) \end{cases} \quad (26)$$

where Φ *is defined by (23) and* π^Q_Θ *is the scaled vector projection with respect to the set* Θ.

Proof Consider the adaptive control law (26). From Theorem 2.8, one has that the given parameter update law:

$$\dot{p} = \pi^Q_\Theta(p, \Phi(x,w,u)),$$

insures $p(t) \in \Theta$ and

$$(p - \theta)^T Q(\dot{p} - \Phi(x,w,u))$$

$$= (p-\theta)^T Q(\pi^Q_\Theta(p, \Phi(x,w,u)) - \Phi(x, w, u)) \leq 0,$$

which implies

$$\int_0^T (p(t) - \theta)^T Q(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt \leq 0$$

for all $T \in \mathbb{R}^+$. Now apply the adaptive control law (26), we have the relation (3.2), which implies

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt + (p(0)-\theta)^T Q(p(0)-\theta),$$

for all $T \in \mathbb{R}^+$.

The direct parameter projection method can be also applied.

Theorem 3.5 **(Adaptive $\mathcal{H}_\infty$-Control with Direct Parameter Projection)** *Consider the parameter-dependent system (12). Suppose there exists a non-negative function* $V : \mathbb{R}^n \to \mathbb{R}^+$ *such that for each* $\theta \in \Theta$, *(3.2) is satisfied. Then the adaptive* $\mathcal{H}_\infty$*-control problem has a solution with a* positive *definite matrix* $Q \in \mathbb{R}^{r\times r}$. *And an adaptive control law* is given *by*

$$\begin{cases} \dot{\pi} = \Phi(x, w, -\frac{1}{4} g_2^T(x,p)\frac{\partial V^T}{\partial x}(x)) \\ p = \Pi^Q_\Theta(\pi) \\ u = -\frac{1}{2} g_2^T(x,p)\frac{\partial V^T}{\partial x}(x) \end{cases} \quad (27)$$

where Φ *is defined by (23) and* Π^Q_Θ *is the scaled direct parameter projection with respect to the set* Θ.

Proof Consider the adaptive control law (27). Suppose $p(t)$ for $t \in [0, \infty)$ is generated by the resulting update law:

$$\dot{\pi} = \Phi(x, w, u),$$

and

$$p(t) = \Pi^Q_\Theta(\pi(t));$$

then from Theorem 2.6, one has $p(t) \in \Theta$ and for all $T \in \mathbb{R}^+$,

$$\int_0^T (p(t) - \theta)^T Q(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt$$

$$= \int_0^T (p(t) - \theta)^T Q(\dot{p}(t) - \dot{\pi}(t))dt \leq 0.$$

Now apply the adaptive control law (27), we have the relation (3.2), which implies

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt + (p(0)-\theta)^T Q(p(0)-\theta),$$

for all $T \in \mathbb{R}^+$.

Remark 3.6 *It is interesting to compare Theorem 9.5 with the sufficient condition result for the minimax adaptive problem* in *[2]. As* in the *above theorem, the sufficient* condition *in [2] for the* minimax *adaptive control problem to have solution* is *that there exists a non-negative function* $V : \mathbb{R}^n \to \mathbb{R}^+$ *independent of the parameter such that the Hamilton-Jacobi inequality (3.2) is satisfied; and the control action u is also obtained by the parameter projection. However, the implications of the function V(z)* in *the two papers are different. In this paper, V(z) is just the storage function of the parameterized* $\mathcal{H}_\infty$ *-control system, but not the resulting adaptive* $\mathcal{H}_\infty$ *-control system, In [2], V(x)* is *the storage function of the resulting minimax adaptive control system,* i.e., *an upper bound of the value junction. The adaptive controller in this paper is simpler than that* given in [2]. *However, the use and* implications *of* parameter *projection* in *[2] and this paper are different. The scaling matrix of parameter projection* in *[2]* is *dependent on state,* while *the projection in this paper is constant* in *this sense.*

In conclusion, it is noted that the direct parameter projection guarantees that the adaptive control law (27) is continuous, while the adaptive control law (26) using vector projection is not. The latter control law can be made continuous using the vector projection defined by (9) under some smoothness assumption,

Acknowledgement

This research was performed at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

References

[1] J.-P. Aubin, *Viability Theory,* Boston, MA: Birkhauser, 1991.

[2] G. Didinsky and T. Basar, "Minimax Adaptive Control of Uncertain Plants," *Proc. of 33rd IEEE CDC,* Orlando, FL, 1994.

[3] J.C. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems", *IEEE T-AC,* Vol.34, pp.831 -847, 1989.

[4] G.C. Goodwin and D.Q. Mayne, "A Parameter Estimation Perspective of Continuous Time Model Reference Adaptive Control," *Automatica,* Vol.23(1), pp.57–70, 1987.

[5] P.A. Ioannou and J. Sun, *Robust Adaptive Control,* Upper Saddle River, NJ: PTR Prentice-Hall, 1996.

[6] I. Kanellakopoulos, P. Kokotovic, and R. Marine, "An 'Extended Direct Scheme for Robust Adaptive Nonlinear Control," *Automatica,* Vol.27(2), pp.247–255, 1991.

[7] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design, 1995.*

[8] W.-M. Lu and J.C. Doyle, "$\mathcal{H}_\infty$ Control of Nonlinear Systems: A Class of Controllers," *Caltech CDS Tech. Memo.,* No. *CIT-CDS-93-008,* also *Proc. 1993 IEEE CDC, San* Antonio, TX, 1993.

[9] W.-M. Lu and A. Packard, '&Adaptive $\mathcal{H}_\infty$-Control for Nonlinear Systems: A Dissipation Theoretical Approach," *Proc. 1997 A CC, Al*buquerque, NM, June, 1997.

[10] W.-M. Lu, F. Hadaegh, and A. Packard, "Adaptive Robust Control of Nonlinear Uncertain Systems with Structured Uncertainty," *1997 IEEE CDC* (submitted), San Diego, CA.

[11] D.G. Luenberger, *Optimization by Vector Space Methods,* New York: John Wiley& Sons, Inc., 1969.

[12] Z. Pan and T. Basar, "Adaptive Controller Design for Tracking and Disturbance Attenuation in Parameter-Strict-Feedback Nonlinear Systems," *IEEE T-AC* (submitted), 1996.

[13] P.C. Parks, "Lyapunov Redesign of Model Reference Adaptive Control Systems," *IEEE T-AC,* Vol.11, pp.362-367, 1966.

[14] M.M. Polycarpou and P.A. Ioannou, "On the Existence and Uniqueness of Solutions in Adaptive Cent rol Systems," *IEEE T-AC,* Vol.38(3), pp.474–479, *1993.*

[15] J-B. Pomet and L. Praly, "Adaptive Nonlinear Regulation: Estimation from the Lyapunov Equation," *IEEE T-AC,* Vol.37(6), *pp.729-740, 1992.*

[16] L. Praly, G. Bastin, J.-B. Pomet, and Z.P. Jiang, "Adaptive Stabilization of Nonlinear Systems," *Foundations of Adaptive Control* (P.V. Kokotovic, cd), Berlin: Springer-Verlag, pp.347-433, 1991.

[17] S. Rangan and K. Poolla, "Adaptive $\mathcal{H}_\infty$ Control with Multiple Nonlinear Models," *Preprint, 1996.*

[18] R.T. Rockafellar, *The Theory of Subgradients and Its Applications to Problems of Op*timization: *Convex and Nonconvex Functions,* Berlin: Heldermann Verlag, 1981.

[19] N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method, New* York: Springer-Verlag, 1977.

[20] D.G. Taylor, P.V. Kokotovic, R. Marine, and [. Kanellakopoulos, "Adaptive Regulation of Nonlinear Systems with Unmodeled Dynamics," *IEEE T-AC,* Vol.34(4), pp.405–412, 1989.

[21] A.J. van der Schaft, "$\mathcal{L}_2$-Gain Analysis of Nonlinear Systems and Nonlinear State Feedback $\mathcal{H}_\infty$-Control", *IEEE Trans. AC,* Vol. AC-37, pp.770–784, 1992.

[22] J.C. Willems, "Dissipative Dynamical Systems," (Parts I and 11), *Arch. Rat. Mech. Anal.,* vol.45, pp.321-393, *1972.*

[23] X.H. Yang, F. Wu, and A. Packard, "Adaptive Control of Full Information Problem," *Proc. 1995 ACC,* Seattle, WA, pp.3371-3372, 1995.