Design of Serially Concatenated Interleaved Codes

S. Benedetto, D. Divsalar, G. Montorsi, F. Pollara

Abstract — A serially concatenated code with interleaver consists of the cascade of an outer encoder, an interleaver permuting the outer codeword bits, and an inner encoder whose input words are the permuted outer codewords. In this paper we derive design guidelines for the outer and inner codes that maximize the interleaver gain and the asymptotic slope of the error probability curves.

Keywords --- Concatenated codes, Turbo codes, Parallel and serial concatenation of codes.

I. Introduction

In his goal to find a class of codes whose probability of error decreased exponentially at rates less than capacity, while decoding complexity increased only algebraically, David Forney [1] arrived at a solution consisting of the multilevel coding structure known as *concatenated code.* It consists of the cascade of an *inner* code and an *outer* code, which, in Forney's approach, would be a relatively short inner code (typically, a convolutional code) admitting simple maximum-likelihood decoding, and a long high-rate algebraic nonbinary Reed-Solomon outer code equipped with a powerful algebraic error-correction algorithm, possibly using reliability information from the inner decoder. An interleaver is sometimes used between the two encoders to separate bursts of errors produced by the inner decoder.

We find then, in a "classical" concatenated coding scheme, the main ingredients that formed the basis for the invention of "turbo codes" [2], namely two, or more, *constituent* codes (CCs) and an *interleaver.* In the following, we will refer to turbo codes as *parallel concatenated convolutional codes* (PCCCs).

In this paper, we consider the serial concatenation of interleaved codes or *serially concatenated codes* (SCCs), called SCBC or SCCC according to the nature of CCS, that can be block (SCBC) or convolutional codes (SCCC). For this class of codes, analytical upper bounds to the performance of a maximum-likelihood (ML) decoder had been derived in [3] and [4]. Here, we propose design guidelines leading to the optimal choice of CCs that maximize the *interleaver gain* and the asymptotic. code performance.

II. Design of Serially Concatenated Codes with Interleaver

In [4] we proved that the bit error probability of SCBCs using a *uniform* interleave [5] and a maximum-likelihood decoder can be upper bounded as

$$P_b(e) \leq \sum_{w=1}^{k} \frac{w}{k} A^{C_S}(w, H)|_{H=e^{-R_c E_b/N_0}}\ , \qquad (1)$$

where $R_c =$ k/n is the rate of C_S, E_b/N_0 is the bit signal-to-noise ratio, $A^{C_S}(w, H)$ is the *conditional weight enumerating function* (CWEF) of the SCBC

$$A^{C_S}(w, H) = \sum_h A^{C_S}_{w,h} H^h\ ,$$

where $A^{C_S}_{w,h}$ is the number of codewords of the SCBC with weight h associated to an input word of weight, w.

The coefficients $A^{C_S}_{w,h}$ of the CWEF can be obtained from those of the two CCS as

$$A^{C_S}_{w,h} = \sum_{l=0}^{N} \frac{A^{C_o}_{w,l} \times A^{C_i}_{l,h}}{\binom{N}{l}}\ . \qquad (2)$$

where the superscripts C_o and C_i refer to the *outer* and *inner* code, respectively.

For SCCCs (whose block diagram is shown in Fig. 1), computing the upper bound to the bit error probability performance requires the definition of an equivalent block code, formed by the sequences of the SCCC with length NR_c^o that join the zero states of both CCS. Thus, performance evaluation requires the knowledge of the CWEFs $A^{C_o}_{w,l}$ and $A^{C_i}_{l,h}$ of the two CCs and then the application of (2) and (1), respectively.

The bound to the bit error probability can be rewritten as

$$\begin{aligned} P_b(e) &\leq \sum_{w=w_m^o}^{NR_c^o} \frac{w}{NR_c^o} A^{C_S}(w, H)|_{H=e^{-R_c E_b/N_0}} \\ &= \sum_{h=h_m}^{N/R_c^i} \sum_{w=w_m^o}^{NR_c^o} \frac{w}{NR_c^o} A^{C_S}_{w,h} e^{-hR_c E_b/N_0}\ , \end{aligned} \qquad (3)$$

where w_m^o is the minimum weight of an input sequence generating an error event of the outer code, and h_m is the minimum weight of the code words of C_S.

To evaluate the CWEFs of the CCS, consider a rate $R = p/n$ convolutional code C with memory ν, and its equivalent $(N/R,\ N - p\nu)$ block code whose codewords are all sequences of length N/R bits of the convolutional code starting from and ending at the zero state. By definition, the code words of the equivalent block code are concatenations of error events of the convolutional codes. Let

$$A(l, H, j) = \sum_h A_{l,h,j} H^h$$

S. Benedetto and G. Montorsi are with Dipartimento di Elettronica, Politecnico di Torino, C.so Duca degli Abruzzi n.24, 10129 Torino -Italy.

D. Divsalar and F. Pollara are with Jet Propulsion Laboratory, 4800 Oak Grove Drive, Pasadena, CA 91109- USA.

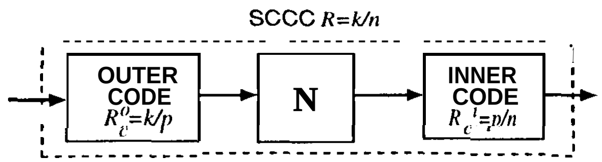

Fig. 1. Serially concatenated (n, k, N) convolutional code.

be the weight enumerating function of sequences of the convolutional code that concatenate j error events with total input weight l, where $A_{l,h,j}$ is the number of sequences of weight h, input weight 1, and number of concatenated error events j. For N much larger than the memory of the convolutional code, the coefficient, $A^C_{l,h}$ of the CWEF of the equivalent block code can be approximated by[1]

$$A^C_{l,h} \sim \sum_{j=1}^{n_M} \binom{N/p}{j} A_{l,h,j} \tag{4}$$

where n_M, the largest number of error events concatenated in a codeword of weight h and generated by a weight 1 input sequence, is a function of h and 1 that depends on the encoder, as we will see later.

Let us return now to the block code equivalent to the SCCC. Using previous result (4) with $j = n^i$ for the inner code, and the analogous one for the outer code[2]

$$A^{C_o}_{w,l} \sim \sum_{n^o=1}^{n^o_M} \binom{N/p}{n^o} A^o_{w,l,n^o}$$

and substituting them into (2), we obtain the coefficient $A^{C_S}_{w,h}$ of the serially concatenated block code equivalent to the SCCC in the form

$$A^{C_S}_{w,h} \sim \sum_{l=d^o_f}^{N} \sum_{n^o=1}^{n^o_M} \sum_{n^i=1}^{n^i_M} \frac{\binom{N/p}{n^o}\binom{N/p}{n^i}}{\binom{N}{l}} A^o_{w,l,n^o} A^i_{l,h,n^i} , \tag{5}$$

where d^o_f is the free distance of the outer code.

We are interested in large inter] eaver lengths, and thus use for the binomial coefficient the asymptotic approximation

$$\binom{N}{n} \sim \frac{N^n}{n!} .$$

Substitution of this approximation in (5) yields

$$A^{C_S}_{w,h} \sim \sum_{l=d^o_f}^{N} \sum_{n^o=1}^{n^o_M} \sum_{n^i=1}^{n^i_M} \frac{N^{n^o+n^i-l} l!}{p^{n^o+n^i} n^o! n^i!} A^o_{w,l,n^o} A^i_{l,h,n^i} . \tag{6}$$

[1] This assumption **permits neglecting the length of error events compared to N, and assuming that the number of ways j input sequences producing j error events can be arranged in a register of length N is** $\binom{N/p}{j}$. The **ratio N/p derives from the fact that the code has rate p/n, and thus N bits correspond to *N/p* input words or, equivalently, trellis steps.**

[2] **In the following, superscripts "o" and "i" will refer to quantities pertaining to outer and inner code, respectively.**

Finally, substituting (6) into (3), gives the bit error probability bound in the form

$$P_b(e) \lesssim \sum_{h=h_m}^{N/R^i_c} e^{-hR_cE_b/N_0} \sum_{l=d^o_f}^{N} \sum_{n^o=1}^{n^o_M} \sum_{n^i=1}^{n^i_M} \frac{N^{n^o+n^i-l-1} l!}{p^{n^o+n^i-1} n^o! n^i!} \cdot \sum_{w=w^o_m}^{NR^o_c} \frac{w}{k} A^o_{w,l,n^o} A^i_{l,h,n^i} . \tag{7}$$

Using expression (7) as the starting point, we will obtain some important design considerations. The bound (7) to the bit error probability is obtained by adding termsof the first summation with respect to the SCCC weights h. 'he coefficients of the exponentials in h depend, among other parameter, on N. For large N, and for a given h, the dominant coefficient of the exponentials in h is the one for which the exponent of N is maximum. Define this maximumexponent as

$$\alpha(h) \triangleq \max_{w,l} \{n^o + n^i - 1 - 1\} .$$

Evaluating $\alpha(h)$ in general is not possible without specifying the CCs. Thus, we will consider two important cases, for which general expressions can be found.

A. The exponent of N for the minimum weight

For large values of E_b/N_0, the performance of the SCC are dominated by the first term of the summation with respect to h, corresponding to the minimum value h = h_m. Remembering that, by definition, n^i_M and n^o_M are the maximum number of concatenated error events in codewords of the inner and outer code of weights h_m and l, respectively, the following inequalities hold true:

$$n^i_M \leq \left\lfloor \frac{h_m}{d^i_f} \right\rfloor , \quad n^o_M \leq \left\lfloor \frac{l}{d^o_f} \right\rfloor$$

and

$$\alpha(h_m) \leq \left\lfloor \frac{h_m}{d^i_f} \right\rfloor + \left\lfloor \frac{l_m(h_m)}{d^o_f} \right\rfloor - l_m(h_m) - 1 , \tag{8}$$

where $l_m(h_m)$ is the minimum weight l of codewords of the outer code yielding a codeword of weight h_m of the inner code, and $\lfloor x \rfloor$ mean s''integer part of x".
In most cases, $l_m(h_m) < 2d^o_f$, and $h_m < 2d^i_f$, so that $n^i_M = n^o_M = 1$, and (8) becomes

$$\alpha(h_m) = 1 - l_m(h_m) \leq 1 - d^o_f . \tag{9}$$

'l'he result (9) shows that the exponent of N corresponding to the minimum-weight of SCCC codewords is always negative for $d^o_f \geq 2$, thus yielding an interleaver gain at high E_b/N_0. Substitution of the exponent $\alpha(h_m)$ into *(7)* truncated to the first term of the summation with respect to h yields

$$\lim_{\frac{E_b}{N_0} \to \infty} P_b(e) \lesssim B_m N^{1-d^o_f} \exp(-h_m R_c E_b/N_0) \tag{lo}$$

where B_m is a suitable constant.

Expression (1 O) suggests the following conclusions:

- For the values of E_b/N_0 and N where the SCCC performance is dominated by its free distance $d_f^{Cs} = h_m$, increasing the interleaver length yields a gain in perfor- In au ce.
- To increase the interleaver gain, one should choose an outer code with large d_f^o.
- To improve the performance with E_b/N_0, One should choose au inner and outer code combination such that h_m is large.

These conclusions do not depend on the structure of the CCs, and thus they yield for both recursive and non recursive encoder.

We evaluate then the largest exponent of N, defined as

$$\alpha_M \stackrel{\triangle}{=} \max_h\{\alpha(h)\} = \max_{w,l,h}\{n^o + n^i - 1 - 1\} \ . \qquad (11)$$

This exponent will permit to find the dominant contribution to the bit error probability for N + cm.

B. *The maximum exponent of N*

We need to treat the cases of nonrecursive and recursive inner encoders separately. As we will see, non recursive encoders and block encoders show the same behavior.

A. Block and nonrecursive convolutional inner encoders

Consider the inner code and its impact on the exponent of N in (1 1). For a nonrecursive inner encoder, we have $n_M^i = l$. In fact, every input sequence with weight, one generates a finite-weight error event, so that au input sequence with weight, l will generate, at most, 1 error events corresponding to the concatenation of 1 error events of input weight one. Since the uniform interleaver generates all possible permutation of its input sequences, this event will certainly occur.

Thus, from (11) we have

$$\alpha_M = n_M^o - 1 \geq 0 \ ,$$

and interleaving gain is not allowed. 'l'his conclusion holds true for both SCCC employing nonrecursive inner encoder and for all SCBCs, since block codes have codewords corresponding to input words with weight equal to one.

For those SCCS we *always* have, for some h, coefficients of the exponential in h of (7) that increase with N, and this explains the divergence of the bound arising, for each E_b/N_0, when the coefficients increasing with N become dominant.

B. Recursive inner encoders

In [6], we proved that, for recursive convolutional encoders, the minimum weight of input sequences generating error events is 2. As a consequence, au input sequence of weight, l can generate at most $\lfloor \frac{l}{2} \rfloor$ error events.

Assuming that the inner encoder of the SCCC is recursive, the maximum exponent of N in (11) becomes

$$\begin{aligned}\alpha_M &= \max_{w,l}\left\{n_M^o + \left\lfloor \frac{l}{2} \right\rfloor - l - 1\right\} \\ &= \max_{w,l}\left\{n_M^o - \left\lfloor \frac{l+1}{2} \right\rfloor - 1\right\} \quad . \qquad (12)\end{aligned}$$

The maximization involves l and w, since n_M^o depends on both quantities. In fact, remembering its definition as the maximum number of concatenated error events of codewords of the outer code with weight l generated by input words of weight w, it is straightforward to obtain

$$n_M^o = \min\left[\left\lfloor \frac{w}{w_m} \right\rfloor, \left\lfloor \frac{l}{d_f^o} \right\rfloor\right] \leq \left\lfloor \frac{l}{d_f^o} \right\rfloor \ . \qquad (13)$$

Substituting now the last inequality (13) into (12) yields

$$\alpha_M \leq \max_l\left\{\left\lfloor \frac{l}{d_f^o} \right\rfloor - \left\lfloor \frac{l+1}{2} \right\rfloor - 1\right\} \quad . \qquad (14)$$

The maximization of the RHS of (14) is lengthy but straightforward. 'l'he final result is

$$\alpha_M = -\left\lfloor \frac{d_f^o + 1}{2} \right\rfloor \ . \qquad (15)$$

The value (15) of α_M shows that the exponents of N in (7) are always negative integers. '1'bus, for all h, the coefficients of the exponents in h decrease with N, and we always have an interleaver gain.

Denoting by $d^i_{f,eff}$, as in [5], the minimum weight, of codewords of the inner code generated by weight-2 input sequences, we obtain a different weight $h(\alpha_M)$ for even and odd values of d_f^o.

d_f^o even

For d_f^o *even, the* weight $h(\alpha_M)$ associated to the highest exponent of N, is given by

$$h(\alpha_M) = \frac{d_f^o d^i_{f,eff}}{2} \ ,$$

since it is the weight of au inner codeword that concatenates $d_f^o\ /2$ error events with weight $d^i_{f,eff}$.

Substituting the exponent α_M 'into (7), approximated only by the term of the summation with respect to h corresponding to $h = h(\alpha_M)$, yields

$$\lim_{N\to\infty} P_b(e) \lesssim B_{even} N^{-\frac{d_f^o}{2}} \exp\left[-\frac{d_f^o d^i_{f,eff}}{2} R_c E_b/N_0\right] \ , \qquad (16)$$

where B_{even} is a suiatble constant.

d_f^o odd

For d_f^o odd, the value of $h(\alpha_M)$ is given by

$$h(\alpha_M) = \frac{(d_f^o - 3)d^i_{f,eff}}{2} + h_m^{(3)},$$

where $h_m^{(3)}$ is the minimum weight of sequences of the inner code generated by a weight 3 input sequence. in this case, in fact, we have

$$n_M^i = \frac{d_f^o - 1}{2}$$

concatenated error events, of which $n_M^i - 1$ generated by weight 2 input sequences and one generated by a weight 3 input sequence.

Thus, substituting the exponent α_M into (7) approximated by keeping only the term of the summation with respect to h corresponding to $h = h(\alpha_M)$ yields

$$\lim_{N\to\infty} P_b(e) \lesssim B_{odd} N^{-\frac{d_f^o+1}{2}} \cdot \exp\left\{-\left[\frac{(d_f^o - 3)d^i_{f,eff}}{2} + h_m^{(3)}\right] R_c E_b/N_0\right\}, \quad (17)$$

where B_{odd} is a suitable constant.

In both cases of d_f^o even and odd, we can draw from (16) and (17) a few important design considerations:

- in contrast with the case of block codes and nonrecursive convolutional inner encoders, the use of a recursive convolutional inner encoder always yields an interleave gain. As a consequence, the first design rule states that the **inner encoder must be a convolutional recursive encoder.**
- The coefficient $h(\alpha_M)$ that multiplies the signal-to-noise ratio E_b/N_0 in (7), increases for increasing values of $d^i_{f,eff}$. Thus, we deduce that **the effective free distance of the inner code must be maximized.** Both this and the previous design rule had been stated also for PCCCs[3] [6]. As a consequence, the recursive convolutional encoders optimized for use in PCCCs (see Tables in [6;7]) can be employed altogether as inner CC in SCCCs.
- The interleave gain is equal to $N^{-\frac{d_f^o}{2}}$ for even values of d_f^o and to $N^{-\frac{d_f^o+1}{2}}$ for odd values of d_f^o. As a consequence, we **should choose,** compatibly with the desired rate R_c of the SCCC, **an outer code with a large and, possibly, odd value of the free distance.**
- As to other outer code parameters, N_f^o and $w_{M,f}$ should be minimized. In other words, we should have the minimum number of input sequences generating free distance error events of the outer code, and their input weights should be minimized. Since nonrecursive encoders have error events with w = 1, and, in general, less input errors associated with error events at free distance [8], it can be convenient to **choose as outer code a nonrecursive encoder** with minimum N_f^o and $w_{M,f}$. Conventional nonrecursive convolutional codes found in books (see for example [9]) are appropriate.

[3]For PCCCs, however, both CCs had to comply with those design rules.

C. Examples confirming the design rules

To confirm the design rules obtained asymptotically, i.e. for large signal-to-noise ratio and large interleaver lengths N, we evaluate the upper bound (7) to the bit error probability for two SCCCs, with different interleaver lengths, and compare their performance with those predicted by the design guidelines.

The two SCCCs are obtained as follows: the first, SCCC1, is a (3,1 ,N) SCCC using as outer code a 4-state, (2,1) nonrecursive, convolutional encoder, and as inner code a 4-state, (3,2) recursive, systematic convolutional encoder. The second, SCCC2, is a (3,1 ,N) SCCC, using as outer code a 4-state, (2, 1) recursive, systematic convolutional encoder, and as inner code a 4-state, (3,2) nonrecursive convolutional encoder. The outer, inner, and SCCC code parameters introduced in the design analysis are listed in Table 2. In this table, the CCS are identified through the description of Table 1.

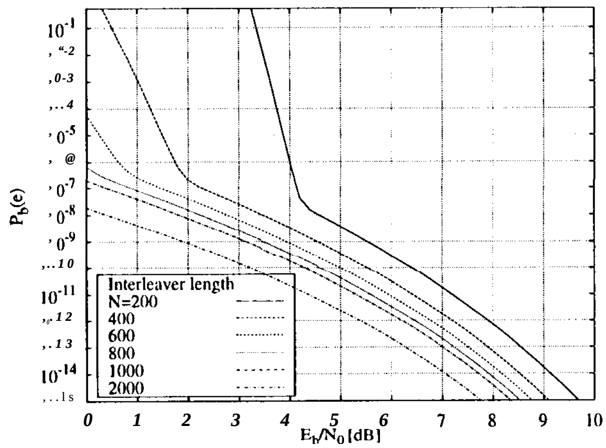

Fig. 2. Analytical bounds for SCCC1

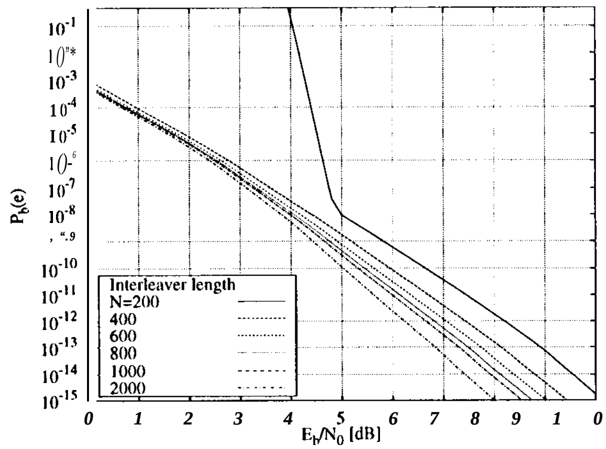

Fig. 9. Analytical bounds for SCCC2

In Figs. 3,2 we plot the bit error probability bounds for

Code description	G(D)
Rate 1/2 NR	$\left[\, 1 + D + D^2,\ 1 + D^2 \,\right]$
Rate 2/3 R	$\begin{bmatrix} 1, & 0, & \frac{1+D^2}{1+D+D^2} \\ 0, & 1, & \frac{1+D}{1+D+D^2} \end{bmatrix}$
Rate 2/3 NR	$\begin{bmatrix} 1+D, & D, & 1 \\ 1+D, & 1, & 1+D \end{bmatrix}$

Table 1. **Generating matrices for the constituent convolutional codes**

Code	Outer code			Inner code			
	Code	w_m^o	d_f^o	Code	w_m^i	d_f^i	$d_{f,eff}^i$
SCCC1	1/2 NR	1	5	2/3 R	2	3	4
SCCC2	1/2 R	2	5	2/3 NR	1	3	4

Code	SCCC			
	h_m	$\alpha(h_m)$	$h(\alpha_M)$	α_M
SCCC1	5	-4	7	-3
SCCC2	5	-4		

Table 2. Design parameters **of CC. and SCCCs for two SCCCs**

SCCCs 1 and *2* of Table 2, with interleaver lengths N = 200,400,600,800,1000,2000.

Consider first SCCC 1, which employs as inner CC a recursive, convolutional encoder as suggested by the design rules and as outer encoder a nonrecursive encoder. Code SCCC1 has d_f^o = 5; thus, from (1 5), we expect an interleaver gain behaving as N^{-3}. This is fully confirmed by the curves of Fig. 2, which, for a fixed and sufficiently large signal-to-noise ratio, show a decrease in $P_b(e)$ of a factor 1000, when N increases from 200 to 2000.

Consider then code SCCC2, which differs from SCCC1 in the choice of a nonrecursive inner encoder, with the same parameters but with the crucial difference of w_m^i = 1. Its bit error probability curves arc shown in Fig. 3. They confirm the previous design predictions. We see, in fact, that for low signal-to-noise ratios, say below 3 dB, no interleaver gain is obtained. This is because the performance are dominated by the exponent h (α_M), whose coefficient increases with N. On the other hand, for larger signal-to-noise ratios, where the dominant contribution to P_b (e) is the exponent with lowest value h_m, the interleave gain makes its appearance. From (9), we foresee a gain behaving as N^{-4}, meaning 4 orders of magnitude for N increasing from 200 to 2000. Curves in Fig. 3 show a smaller gain (slightly higher than 1/1000), which is on the other hand rapidly increasing with E_b/N_0.

111. Conclusions

We have presented design criteria to select constituent codes for constructing serially concatenated codes with interleaver, a concept building on classical concatenated codes and parallel concatenated codes known as "turbo codes". Based on analytical upper bounds to the bit error probability asymptotic in the interleaver length N, design guidelines have identified the crucial parameters for the outer and inner codes that maximize the *interleaver gain* and the asymptotic slope of the error probability curves. The analysis showed that the interleave gain, defined as the factor that decreases the bit error probability as a function of the inter] eaver size, can be made significally higher than for turbo codes.

IV. Acknowledgment

The research in this article has been partially supported by NATO under Research Grant CRG 951208. Sergio Benedetto and Guido Montorsi also acknowledge the support of Italian National Research Council (CNR) under " Progetto Finalizzato Trasporti (Prometheus)", and of MU RST (Progetto 40% " Comunicazioni con Mezzi Mobil i')). The research described in this paper was partially carried out at the Jet l'repulsion Laboratory, California Institute of Technology, under contract, with National Aeronautics and Space Administration (NASA).

References

[1] G.D. Forney Jr., *Concatenated Codes,* M. I. T., Cambridge, MA, 1966.

[2] Claude Berrou, Alain Glavieux, and Punya Thitimajshima, "Near Shannon Limit Error-Correcting Coding and Decoding: Turbo-Codes", in *Proceedings of ICC'93*, Geneve, Switzerland, May 1993, pp. 1064-1070.

[3] Sergio Benedetto and Guido Montorsi, "Serial concatenation of block and convolutional codes", *Electronics Letters*, vol. 32, no. 10, pp. 887-888, May 1996.

[4] Sergio Benedetto and Guido Montorsi, "Serial concatenation of interleaved codes: analytical performance bounds" , in *Proceedings of GLOBECOM '96, London,* UK, Nov. 1996.

[5] Sergio Benedetto and Guido Montorsi, "Unveiling turbo-codes: some results on parallel concatenated coding schemes", *IEEE 'Transactions 071 Information Theory,* vol. **42, no. 2, pp. 409-429,** Mar. 1996.

[6] Sergio Benedetto and Guido Montorsi, "Design of Parallel Concatenated Convolutional Codes", *IEEE Transactions on Communications,* vol. 44, no. 5, PP. 591-600, May 1996.

[7] Dariush Divsalar and Robert J. McEliece, "Effective free distance of turbo codes", *Electronics letters,* vol. 32, no. 5, Feb. 1996.

[8] Punya Thitimajshima, *Systematic recursive convolutional codes and their application to parallel concatenation,* PhD thesis, Université de Bretagne Occidental) Dec. 1993, (in french).

[9] Sergio Benedetto, Ezio Biglieri, and Valentino Castellani, *Digital Transmission Theory,* Prentice-Hall, New York, 1987.