A Continuum Model of the Dynamics of Coupled Oscillator Arrays for Phase Shifterless Beam-Scanning

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Abstract - **The behavior of arrays of coupled oscillators has been previously studied by computational solution of a set of non-linear differential equations describing the time dependence of each oscillator in the presence of signals coupled from neighboring oscillators. The equations are sufficiently complicated that intuitive understanding of the phenomena which arise is exceedingly difficult. We propose a simplified theory of such arrays in which the relative phases of the oscillator signals is represented by a continuous function defined over the array. This function satisfies a linear partial differential equation of diffusion type which may be solved via the Laplace transform. This theory is used to study the dynamic behavior of a linear array of oscillators which results when the end oscillators are detuned to achieve the phase distribution required for steering a beam radiated by such an array.**

I. INTRODUCTION

It has been suggested that an array of coupled oscillators can be used to control the phase distribution across the aperture of an array antenna in such a manner as to effect steering of the beam without the use of phase shifters.[1] Such an array is illustrated schematically in Figure 1. The behavior of such arrays of oscillators has been described in detail using a coupled set of non-linear differential equations.[2][3] These equations are derived by first describing the behavior of an individual oscillator with injection locking in the manner of Adler [4] and then allowing the injection signals to be provided by the neighboring oscillators in the array. Pogorzelski [5] noted that the resulting formalism contains a matrix operator resembling a discretized version of the familiar Laplacian operator and conjectured that, as a consequence, solutions of Laplace's equation may play a significant role in the behavior of the array. In exploring the consequences of this conjecture, York showed in [6] that, in steady state, a correct approximate description emerges as Poisson's equation in which the distribution of the free running frequencies of the oscillators appears as a source term analogous to charge density in electrostatics.[7] A related approach, where a discrete array is modeled as a continuum of oscillators governed by a single global differential equation, has been described by Rand, Cohen, and Holmes [8] in a mathematical description of certain biological systems. Interestingly, their equations are virtually identical in form to those described in [6].

Reducing the problem to that of solving Poisson's equation is remarkably useful. Considerable insight into the operation of these arrays can be obtained by analogy to the corresponding electrostatic problem. For example, previously reported beamscanning techniques [1], which are difficult to explain intuitively, are reduced to an equivalent parallel-plate capacitor problem. In addition, since the system is described by a single differential equation, a new spectrum of analytical tools can be brought to bear on the problem, and this allows us to quantify both the steady state and the dynamic behaviors of the array in new ways that increase our understanding of coupled-oscillator systems. Lastly, although not treated here, the continuum analysis can be generalized to two-dimensional arrays, resulting in a considerable computational advantage for large 2D array systems.

In this paper, we focus on development of the continuum model and associated boundary conditions, with application to the analysis of time-dependent phase relationships in linear oscillator chains. In particular, a description of beam-settling time in a scanning application is obtained. Laplace transform techniques are used for the transient analysis, leading to analytic solutions for the phase evolution. We show that the phases evolve on a time scale that is related to the size of the array and the coupling strength, and remain "well-behaved" in the sense of maintaining a well-defined beam pattern during the transient period.

II. DERIVATION OF THE CONTINUUM MODEL

Analysis of an array of 2N+1coupled oscillators is embodied in the equation,

$$\frac{d\theta_i}{dt} = \omega_{tune,i} - \sum_{j=-N}^{N} \Delta\omega_{lock,ij} \sin(\Phi_{ij} + \theta_i - \theta_j) \tag{1}$$

for i=-N, -N+1...0, 1, 2, ...N, which is in essence equation (12) of reference [2] repeated here for convenience. Φ_{ij} is the phase of the injection signal from oscillator j evaluated at oscillator i, the coupling phase, and ε_{ij} is the amplitude of this signal while the inter-oscillator locking range is defined by,

$$\Delta\omega_{lock,ij} = \frac{\varepsilon_{ij}\omega_{tune,i}}{2Q}\frac{\alpha_j}{\alpha_i} \tag{2}$$

where Q is the quality factor of the oscillators and $\omega_{tune,i}$ is the free running frequency of the i[th] oscillator. α_i is the amplitude of the output signal of the i[th] oscillator. The phase, θ_i, is the phase of the i[th] oscillator; that is,

$$\theta_i = \omega_{ref} t + \phi_i \tag{3}$$

where ω_{ref} is the reference frequency for defining the phase of each oscillator. Applying this to a one dimensional array and following [2], we assume only nearest neighbor coupling, zero coupling phase, and that all of the inter-oscillator locking ranges are identical. This leads to,

$$\frac{d\theta_i}{dt} = \omega_{tune,i} - \Delta\omega_{lock} \sum_{\substack{j=i-1 \\ j\neq i}}^{j=i+1} \sin(\theta_i - \theta_j) \tag{4}$$

which is equation (17) of [2] under the assumption of zero coupling phase. This will be the starting point for the present derivation of the continuum model.

Assuming that the inter-oscillator phase differences are small, we approximate the sine function by its argument thus obtaining,

$$\frac{d\theta_i}{dt} = \omega_{tune,i} - \Delta\omega_{lock} \sum_{\substack{j=i-1 \\ j\neq i}}^{j=i+1} (\theta_i - \theta_j) = \omega_{tune,i} - \Delta\omega_{lock}(\theta_i - \theta_{i+1} + \theta_i - \theta_{i-1}) \tag{5}$$

which, using (3), can be re-written in the form,

$$\frac{d\phi_i}{dt} = \omega_{tune,i} - \omega_{ref} + \Delta\omega_{lock}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \tag{6}$$

for i=-N, -N+1, ... 0, 1, 2, ...N. At this point we note that the quantity in parentheses is merely a finite difference approximation for the second derivative of the phase with respect to a spatial variable, x, which corresponds to the index, i, at integer values. Thus, (6) can now be easily recognized as the finite difference approximation corresponding to the partial differential equation,

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial\phi}{\partial\tau} = -\frac{\omega_{tune} - \omega_{ref}}{\Delta\omega_{lock}} \tag{7}$$

for $-a-\frac{1}{2} \le x \le a+\frac{1}{2}$ where $\phi(x,\tau)$ is the phase across the array and the unitless time, τ, is the time, t, multiplied by the locking range $\Delta\omega_{lock}$. The points $x=\pm a$ correspond to the index values $i=\pm N$. The array is taken to extend over 2N+1 unit cells with an oscillator at the center of each unit cell. This leads to the range of x noted above. Note that the driving function is the distribution of the oscillator free running (tuning) frequency of each oscillator relative to the reference frequency. Averaging (7) over the length of the array results in,

$$<\frac{\partial^2\phi}{\partial x^2}> - \frac{\partial <\phi>}{\partial\tau} = -\frac{<\omega_{tune}> - \omega_{ref}}{\Delta\omega_{lock}} \tag{8}$$

The first term is zero because the integral of the second derivative of the phase over the length of the array is equal to the difference of the phase gradients evaluated at the endpoints and they are both zero by virtue of the Neumann boundary condition there. Now, by definition the instantaneous frequency of the oscillators is given by,

$$\frac{\omega}{\Delta\omega_{lock}} = \frac{d\phi}{d\tau} + \frac{\omega_{ref}}{\Delta\omega_{lock}} \tag{9}$$

Substituting (9) into (8), we have that,

$$<\omega> = \frac{<\omega_{tune}>}{\Delta\omega_{lock}} \tag{10}$$

Recognizing that in steady state all of these mutually locked oscillators will oscillate at the same frequency, we can interpret this to imply that this ensemble frequency is merely the average of the tuning frequencies, a result which has been previously obtained using the discrete model of such an array.[2] In the examples to follow, it will thus be convenient to set the reference frequency equal to the initial value of this ensemble frequency.

A. The Infinite Length Array

Consider now an array for which a=∞; that is, a linear array of infinite length. Let the oscillator at x=b be detuned by an amount C (measured in locking ranges) from the ensemble frequency at t=0. This implies that the driving function for (7) may be represented by,

$$-\frac{\omega_{tune} - \omega_{ref}}{\Delta\omega_{lock}} = -Cu(\tau)\delta(x-b) \tag{11}$$

(Note that while it is not limited in this linearized theory, C is in reality limited by the fact that the magnitude of the sine function in (4) must be less than or equal to unity.) The Laplace transform of (7) with respect to τ is then,

$$\frac{\partial^2 f}{\partial x^2} - sf = -\frac{C}{s}\delta(x-b) \tag{12}$$

where $f(x,s)$ is the Laplace transform of $\phi(x,\tau)$ and $\phi(x,0^+)$ is taken to be zero for simplicity. The resulting solution for the transform of the phase is,

$$f(x,s)=\frac{C}{2s\sqrt{s}}e^{-|x-b|\sqrt{s}} \tag{13}$$

Which is recognized as -C times the Green's functions for equation (12). The inverse Laplace transform is then,

$$\phi(x,\tau)=C\left[2\sqrt{\frac{\tau}{\pi}}e^{-\frac{(x-b)^2}{4\tau}}-|x-b|\,erfc\left(\frac{|x-b|}{2\sqrt{\tau}}\right)\right]u(\tau) \tag{14}$$

The behavior of this function over the ranges $-10\leq x\leq 10$ and $0<\tau<250$ is shown in Figure 2. Note that at infinite time, this function diverges as the square root of the time. That is, the phase never reaches a steady state value. However, differentiating this function with respect to time gives the dynamic behavior of the frequency in the form,

$$\omega(x,\tau)=\omega_{ref}+\Delta\omega_{lock}\frac{C}{\sqrt{\pi\tau}}e^{-\frac{(x-b)^2}{4\tau}}u(\tau) \tag{15}$$

which at infinite time converges to the steady state value equal to the original ensemble frequency, ω_{ref}, as one over the square root of the time. This is a manifestation of the fact that changing the tuning of one oscillator in an infinite array does not change the ensemble frequency. This frequency distribution is shown in Figure 3.

At this point, one might question the use of the Dirac delta function to represent the spatial distribution of the detuning in (11) preferring instead the use of a unit amplitude square pulse one unit cell wide. The result of using this alternate representation can be readily obtained from (14) by numerical convolution and may thus be seen to differ very little from (14) itself. The greatest difference occurs at x=b and the two results at this point are displayed as a function of time in Figure 4 for comparison. We choose in this treatment to use the delta function representation for analytical convenience.

B. The Finite Length Array

Consider now an array extending from -a to a in x thus having 2a+1 oscillators. To derive the dynamic behavior of the phase in such an array with the element at x=b detuned, we must effectively add homogeneous solutions of (12) to the particular integral (14) so as to satisfy the boundary conditions at the ends of the array. These boundary conditions are most easily derived via the following artifice. Imagine one additional fictitious oscillator added at each end of the array. Let these additional oscillators each be tuned as a function of time in such a manner as to maintain the oscillator phase equal to the phase of the corresponding actual end oscillator. The resulting injection signals from the fictitious oscillators will then have no effect on the dynamics of the actual array. In this sense, the extended array simulates the actual array. However, since the phases of the fictitious oscillator at each end and the corresponding end oscillator are maintained equal, the phase gradient at the ends of the array is seen to be zero. Therefore, the boundary conditions at the array ends, $x = a + \frac{1}{2}$ and $x = -a - \frac{1}{2}$, are seen to be the classical Neumann conditions independent of time. (It is interesting to note that the necessary tuning of the fictitious oscillators may be obtained from (4) if desired once the solution for the array phase dynamics has been obtained as described below.)

Following the prescription suggested above we postulate a solution of the form,

$$f(x,s) = \frac{C}{2s\sqrt{s}} e^{-|x-b|\sqrt{s}} + C_R e^{-x\sqrt{s}} + C_L e^{x\sqrt{s}} \tag{16}$$

One can then determine the unknown constants, C_R and C_L, by imposing Neumann boundary conditions at the array ends. The inverse Laplace transform is then easily expressed as a residue series over an infinite set of poles located on the negative real axis. However, we have found that it requires considerable experience with the algebra involved to effect simplification of the resulting expressions. Therefore, in this presentation we adopt an alternative approach leading directly to the result in simplified form. We proceed as follows.

Recognizing that we are dealing with a self-adjoint boundary value problem of Sturm-Liouville type, it becomes clear that the Green's function can be directly expressed as a linear combination of the normalized eigenfunctions of the differential operator satisfying the boundary conditions. These eigenfunctions are,

$$u_n = \frac{\sqrt{2}\cosh\left(x\sqrt{s_n}\right)}{\sqrt{2a+1}} \tag{17}$$

$$v_m = \frac{\sqrt{2}\sinh\left(x\sqrt{s_m}\right)}{i\sqrt{2a+1}} \tag{18}$$

where s_n and s_m are given by,

$$\sinh\left[\sqrt{s_n}\left(a+\frac{1}{2}\right)\right] = 0 \tag{19}$$

$$\cosh\left[\sqrt{s_m}\left(a+\frac{1}{2}\right)\right] = 0 \tag{20}$$

That is,

$$s_n = -\left(\frac{2n\pi}{2a+1}\right)^2 \tag{21}$$

for n = 0, 1, 2, ... and

$$s_m = -\left(\frac{(2m+1)\pi}{2a+1}\right)^2 \tag{22}$$

for m=0, 1, 2, ...

The Green's function can thus be immediately written in the form,

$$\tilde{G}(x,x';s) = \sum_{n=0}^{\infty} \frac{2\cosh\left(x'\sqrt{s_n}\right)\cosh\left(x\sqrt{s_n}\right)}{(2a+1)(s_n - s)} - \sum_{m=0}^{\infty} \frac{2\sinh\left(x'\sqrt{s_m}\right)\sinh\left(x\sqrt{s_m}\right)}{(2a+1)(s_m - s)} \tag{23}$$

where the tilde denotes the function in the transform domain. Note that, despite the presence of the square roots, there is no branch cut because the solution is an even function of the square root of s. The solution of (12) is therefore,

$$\phi(x,s) = -\frac{C}{s}\sum_{n=0}^{\infty} \frac{2\cosh\left(b\sqrt{s_n}\right)\cosh\left(x\sqrt{s_n}\right)}{(2a+1)(s_n - s)} + \frac{C}{s}\sum_{m=0}^{\infty} \frac{2\sinh\left(b\sqrt{s_m}\right)\sinh\left(x\sqrt{s_m}\right)}{(2a+1)(s_m - s)} \tag{24}$$

The inverse Laplace transform is now a trivial matter of evaluating the residue at the single pole in each term of the summations and the pole at the origin. (Of course, some care must be taken concerning the double pole at the origin arising from the zero eigenvalue term in the first summation.) The result is,

$$\begin{aligned}\phi(x,\tau) = {} & \frac{C\tau}{2a+1} + C\sum_{n=1}^{\infty} \frac{2\cos\left(b\sqrt{\sigma_n}\right)\cos\left(x\sqrt{\sigma_n}\right)}{(2a+1)\sigma_n} + C\sum_{m=0}^{\infty} \frac{2\sin\left(b\sqrt{\sigma_m}\right)\sin\left(x\sqrt{\sigma_m}\right)}{(2a+1)\sigma_m} \\ & - C\sum_{n=0}^{\infty} \frac{2\cos\left(b\sqrt{\sigma_n}\right)\cos\left(x\sqrt{\sigma_n}\right)}{(2a+1)\sigma_n} e^{-\sigma_n\tau} - C\sum_{m=0}^{\infty} \frac{2\sin\left(b\sqrt{\sigma_m}\right)\sin\left(x\sqrt{\sigma_m}\right)}{(2a+1)\sigma_m} e^{-\sigma_m\tau}\end{aligned} \tag{25}$$

which can also be written in the form,

$$\phi(x,\tau)=\frac{C\tau}{2a+1}$$
$$+C\sum_{n=0}^{\infty}\frac{2\cos\left(b\sqrt{\sigma_n}\right)\cos\left(x\sqrt{\sigma_n}\right)}{(2a+1)\sigma_n}\left(1-e^{-\sigma_n\tau}\right) \tag{26}$$
$$+C\sum_{m=0}^{\infty}\frac{2\sin\left(b\sqrt{\sigma_m}\right)\sin\left(x\sqrt{\sigma_m}\right)}{(2a+1)\sigma_m}\left(1-e^{-\sigma_m\tau}\right)$$

The first term exhibits the shift in the ensemble frequency resulting from step detuning the oscillator at x=b by an amount C. Since the steady state ensemble frequency, which is equal to the average of the tuning frequencies, is now no longer equal to the reference frequency, the phase solution includes a linear time dependence which corresponds to this frequency difference, $\frac{C}{2a+1}$. This may also be seen directly from (7) by averaging the equation over the length of the array which leads to (8). Recall that the first term is zero because of the Neumann boundary conditions. The right hand side is a constant equal to the change in the average tuning frequency; that is, $\frac{C}{2a+1}$. Thus we have,

$$\frac{d<\phi>}{d\tau}=\frac{C}{2a+1} \tag{27}$$

or,

$$<\phi>=\frac{C\tau}{2a+1} \tag{28}$$

The constant of integration is zero by virtue of the initial condition that the initial phase is zero which was implicit in the Laplace transform of the differential equation (12). Thus, we can conclude that, aside from this linear term, the average of the phase over the array remains zero for all time. The behavior of an array with twenty-one oscillators when the oscillator at x=5 is detuned by C is displayed in Figure 5 which is obtained by direct evaluation of (26) suppressing the term linear in time.

The first two summations in (25) are merely Fourier series expressions for the steady state phase distribution and can be evaluated in closed form leading to,

$$\phi_{ss}(x,\tau)=C\sum_{n=1}^{\infty}\frac{2\cos\left(b\sqrt{\sigma_n}\right)\cos\left(x\sqrt{\sigma_n}\right)}{(2a+1)\sigma_n}+C\sum_{m=1}^{\infty}\frac{2\sin\left(b\sqrt{\sigma_m}\right)\sin\left(x\sqrt{\sigma_m}\right)}{(2a+1)\sigma_m}$$
$$=\frac{C}{2(2a+1)}\left[x^2+b^2-(2a+1)|b-x|+\frac{1}{6}(2a+1)^2\right] \tag{29}$$

As will be seen shortly, the quadratic steady state phase is to be expected since it is basically a solution of Poisson's equation with a constant source term.

The slowest exponential decay rate can be found by evaluating (22) for m=0. Thus,

$$\sigma_{\min} \approx \left(\frac{\pi}{2a+1}\right)^2 \tag{30}$$

indicating that the response time of the array under detuning of one oscillator is proportional to the *square* of the number of oscillators (for large a).

If the center oscillator is detuned, (22) is no longer the relevant set of eigenvalues because the corresponding residues are zero by symmetry. In this case, the slowest decay rate is found from the smallest non-zero member of the set (21); that is,

$$\sigma_{\min} \approx \left(\frac{2\pi}{2a+1}\right)^2 \tag{31}$$

which represents a response time which is twice as fast as the nonsymmetric case but, nevertheless, again proportional to the square of the number of oscillators (again, for large a).

Similar results obtain by superposition when multiple oscillators are detuned.

C. The Steady State Solution

The steady state solution derived as a limit of the dynamic solution presented above can also be obtained directly without resorting to the Laplace transformation. This provides a useful analog with electrostatics as will be demonstrated below.

Returning now to equation (7) we note that in steady state the oscillators will all oscillate at the same frequency, ω_0, the average of the tuning frequencies, and the phase will be constant with time. Therefore, $\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial\tau}(\omega_0 t + \phi) = \frac{\omega_0}{\Delta\omega_{lock}} = \Omega_0$ and the differential equation then takes the particularly simple form,

$$\frac{d^2\phi}{dx^2} = -\rho \tag{32}$$

where, $\rho = \Omega_{tune} - \Omega_0$, $\Omega_{tune} = \frac{\omega_{tune}}{\Delta\omega_{lock}}$, and $\Omega_0 = <\Omega_{tune}>$; that is, Poisson's equation of electrostatics where ϕ is the analog of electrostatic potential and ρ is the analog of charge density (times permittivity). This charge density is determined by the deviations of the tuning frequencies from Ω_0 and, as such, its integral over the array is clearly zero implying zero net charge. It is noted at this point that one may determine the tuning necessary to obtain any desired phase distribution along the array by merely substituting the desired phase function into (32) and differentiating twice with respect to x to obtain the tuning function. It will be of particular interest, however, to observe the phase distributions obtainable by tuning only two of the oscillators, those at the two ends of the array

1. One Element Detuned

Assuming that a Green's function G satisfying appropriate boundary conditions is known, the general solution can be written in the form,

$$\phi\,(x)=\phi_o-\int_{-\infty}^{\infty}G(x,x')\,\rho\,(x')\,dx' \tag{33}$$

where ϕ_0 is a constant phase permitting selection of any desired phase reference. It is noted in passing that for an infinitely long array,

$$G(x,x')=\frac{1}{2}|x-x'| \tag{34}$$

We now proceed to construct the analogous solution for the present finite length array extending from $x=-a-\frac{1}{2}$ to $a+\frac{1}{2}$ with the oscillator at x=b detuned by an amount C. That is we wish to solve, subject to Neumann boundary conditions at the endpoints of the array, the equation,

$$\frac{d^2\phi}{dx^2}=\Omega_0-C\delta\,(x-b) \tag{35}$$

This is, of course, equivalent to solving (12) with s=0. As above, the solution can be written as a sum of the same eigenfunctions obtained above with eigenvalues $-\sigma_m$ and $-\sigma_n$ and the result would emerge as exactly the two series in (29) which, as shown above, can be summed in closed form. However, we find it more convenient to obtain the result by direct solution of (35) thus circumventing the series summation step. We proceed as follows. First define solutions to the left and right of the detuned oscillator at b in the form,

$$\phi_L(x)=\frac{1}{2}\Omega_0x^2+\beta_Lx+\gamma_L \quad ;for \quad x\le b \tag{36}$$

$$\phi_R(x)=\frac{1}{2}\Omega_0x^2+\beta_Rx+\gamma_R \quad ;for \quad x\ge b \tag{37}$$

Imposing the Neumann boundary conditions at the ends of the array we find that,

$$\beta_L = \frac{1}{2}\Omega_0(2a+1) \tag{38}$$

$$\beta_R = -\frac{1}{2}\Omega_0(2a+1) \tag{39}$$

Requiring that the solutions have the same value at x=b yields,

$$\gamma_R - \gamma_L = \Omega_0 b(2a+1) \tag{40}$$

so we write,

$$\gamma_L = \gamma - \frac{1}{2}\Omega_0 b(2a+1) \tag{41}$$

$$\gamma_R = \gamma + \frac{1}{2}\Omega_0 b(2a+1) \tag{42}$$

and thus obtain,

$$\phi_L(x) = \frac{1}{2}\Omega_0 x^2 + \frac{1}{2}\Omega_0(2a+1)x + \gamma - \frac{1}{2}\Omega_0 b(2a+1) \tag{43}$$

$$\phi_R(x) = \frac{1}{2}\Omega_0 x^2 - \frac{1}{2}\Omega_0(2a+1)x + \gamma + \frac{1}{2}\Omega_0 b(2a+1) \tag{44}$$

which can be written in the more compact form,

$$\phi(x) = \frac{1}{2}\Omega_0 x^2 - \frac{1}{2}\Omega_0(2a+1)|b-x| + \gamma \tag{45}$$

Now imposing the discontinuity condition on the derivative across x=b,

$$\left.\frac{d\phi}{dx}\right|_{x=b+} - \left.\frac{d\phi}{dx}\right|_{x=b-} = -\Omega_0(2a+1) = -C \tag{46}$$

which gives,

$$\Omega_0 = \frac{C}{2a+1} \tag{47}$$

and this, of course, renders the average "charge density" zero as expected. Finally, using the fact that the average of the phase function over the array is zero we obtain,

$$\gamma = \frac{2a+1}{12} + \frac{b^2}{2(2a+1)} \tag{48}$$

and the final result becomes,

$$\phi(x) = \frac{C}{2(2a+1)}\left[x^2 + b^2 - (2a+1)|b-x| + \frac{(2a+1)^2}{6}\right] \tag{49}$$

in agreement with (29). Recalling, now, that to maintain lock the phase diffference between two adjacent oscillators is limited to $\pi/2$ which limits the phase gradient, one can show from (49) that C is limited to,

$$C < \frac{\pi(2a+1)}{4b} \tag{50}$$

2. End Elements Detuned

We now investigate the aperture phase behavior when only the *end* oscillators are detuned; that is when,

$$\omega(x) = \omega_o + \Delta\omega_L \delta(x+a) + \Delta\omega_R \delta(x-a) \tag{51}$$

By superposition, the result can be immediately written as the sum of two expression of the form (49) with b=a and b=-a. The result is,

$$\phi(x) = \left(\frac{\Delta\omega_L + \Delta\omega_R}{2\Delta\omega_{lock}}\right)\left(\frac{x^2+b^2}{2a+1} + \frac{2a+1}{6}\right) - \left(\frac{\Delta\omega_L - \Delta\omega_R}{2\Delta\omega_{lock}}\right)x \tag{52}$$

From this expression it is clear that the sum of the detunings at the two ends determines the quadratic part of the phase while the difference determines the linear part of the phase. Thus, equal and opposite detuning of the end oscillators produces a linear steady state phase distribution in the aperture and consequently steers the beam. Of course, any detuning of the end oscillators can be resolved into even and odd parts with the even part controlling the quadratic phase and the odd part controlling the linear phase.

III. THE DYNAMICS OF BEAM-STEERING

The above theory can be applied to analyze the behavior of a linear array antenna (Figure 1.) in which the phases of the signals radiated by the elements are controlled by means of a coupled oscillator array. The technique is as described by P. Liao, et. al. [1] in which the outputs from each of the oscillators feeds a radiating element and the end oscillators in the array are detuned in opposite directions to generate a constant phase gradient across the aperture. The complete dynamic solution can be written as the difference of two solutions of the form (26). Each of these corresponds to detuning of the form (51) with unit step time dependence, one solution with b=a and one with b=-a using antisymmetrical tuning; that is, $\Delta\omega_L = -\Delta\omega_R = \Delta\omega_T$. The result is,

$$\phi(x,\tau) = \frac{\Delta\omega_T}{\Delta\omega_{lock}} \sum_{m=0}^{\infty} \frac{2\sin\left(b\sqrt{\sigma_m}\right)\sin\left(x\sqrt{\sigma_m}\right)}{(2a+1)\sigma_m}\left(1-e^{-\sigma_m\tau}\right) \tag{53}$$

This function is shown in Figure 6 and the corresponding far zone radiation pattern of a twenty one element array with half wavelength elements spacing driven with these phases is shown in Figure 7. Note that the pattern retains its basic shape, both main beam and sidelobes, throughout the transient period. The duration of the transient period is governed by the time constant, (30).

IV. CONCLUDING REMARKS

A simplified theory describing the dynamics of coupled oscillator arrays has been presented. It has been shown that the steady state limit of this formalism is analogous to electrostatics and, as such, is governed by Poisson's equation with the electric charge density determined by the tuning of the individual oscillators in the array. The simplified theory has been applied to provide a description of the dynamics of beam steering achieved in the manner proposed by P. Liao, et. al. [1]. That is, the end oscillators of a linear array are detuned in opposite directions from the ensemble frequency resulting in a linear phase variation across the array. A key result is that, in such an arrangement, the slowest time constant governing the dynamics is proportional to the square of the number of elements in the array.

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REFERENCES

1. P. Liao and R. A. York, "A New Phase Shifterless Beam-Scanning Technique Using Arrays of Coupled Oscillators," IEEE Trans. Microwave Theory and Tech., MTT-41, pp. 1810-1815, October 1993.

2. R. A. York, "Nonlinear Analysis of Phase Relationships in Quasi-Optical Oscillator Arrays," IEEE Trans. Microwave Theory and Tech., MTT-41, pp. 1799-1809, October 1993.

3. H.-C. Chang, E. S. Shapiro, and R. A. York, "Influence of the Oscillator Equivalent Circuit on the Stable Modes of Parallel-Coupled Oscillators," IEEE Trans. Microwave Theory and Tech., MTT-45, pp. 1232-1239, August 1997.

4. R. Adler, "A Study of Locking Phenomena in Oscillators," Proc. IEEE, 61, pp.1380-1385, October 1973.

5. Private Communication, cca. February 1996.

6. R. A. York and R. J. Pogorzelski, "Investigation of Phase Control in a Planar Array Feed for a Parabolic Reflector using Coupled Oscillator Techniques," Final Report, JPL Contract No. 960499, November 21, 1996.

7. R. J. Pogorzelski and R. A. York, "A Simplified Theory of Coupled Oscillator Phase Control," IEEE AP-S International Symposium Digest, Montreal, Quebec, Canada, pp. 324-327 , July 1997.

8. R. H. Rand, A.H. Cohen, and P.J. Holmes, "Systems of Coupled Oscillators as Models of Central Pattern generators", Chap 9 of Neural Control of Rythmic Movements in Vertebrates, A.H. Cohen, S. Rossignol, and S. Grillner, eds., Wiley: New York, 1988.

Figure Captions

Figure 1. An array controlled by coupled oscillators.

Figure 2. The dynamic phase behavior of an infinite length linear coupled oscillator array with one oscillator detuned.

Figure 3. The dynamic frequency behavior of an infinite length linear coupled oscillator array with one oscillator detuned.

Figure 4. A comparison of two source representations.

Figure 5. The dynamic behavior of a coupled oscillator array under step detuning of the element at x=5.

Figure 6. The dynamic behavior of a coupled oscillator array under step detuning of the end elements.

Figure 7. The dynamics of beam steering via step detuning of the end elements of a coupled oscillator array.

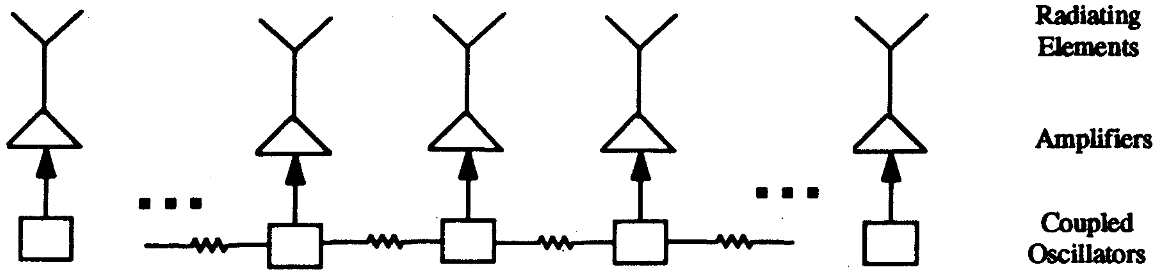

Figure 1.

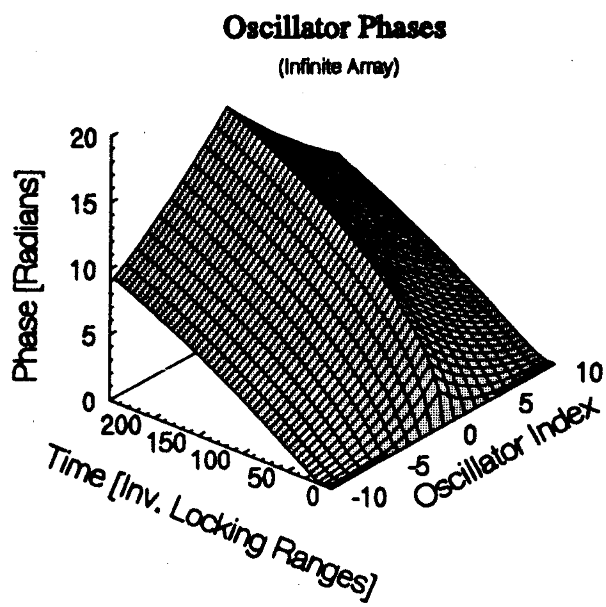

Figure 2.

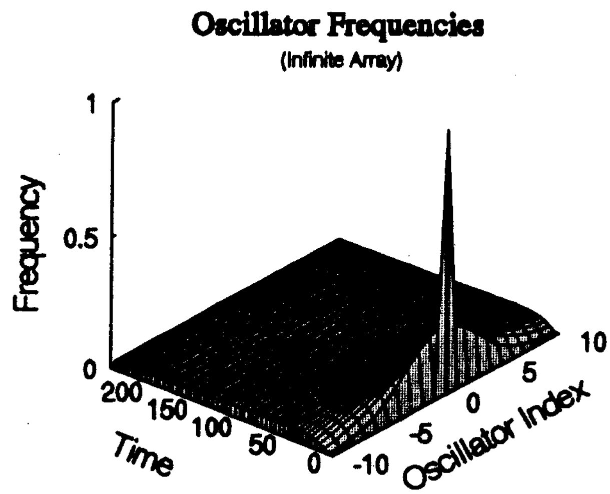

Figure 3.

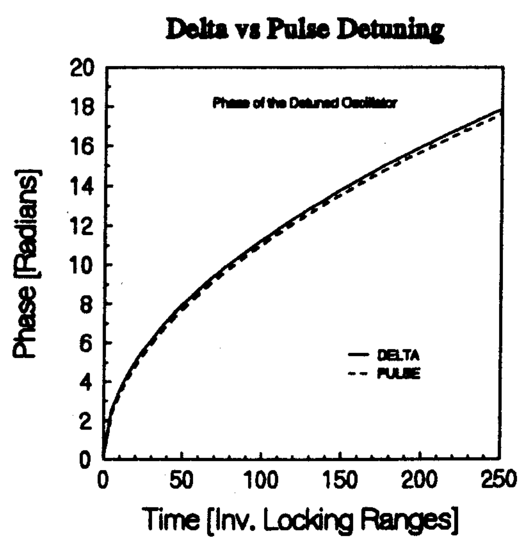

Figure 4.

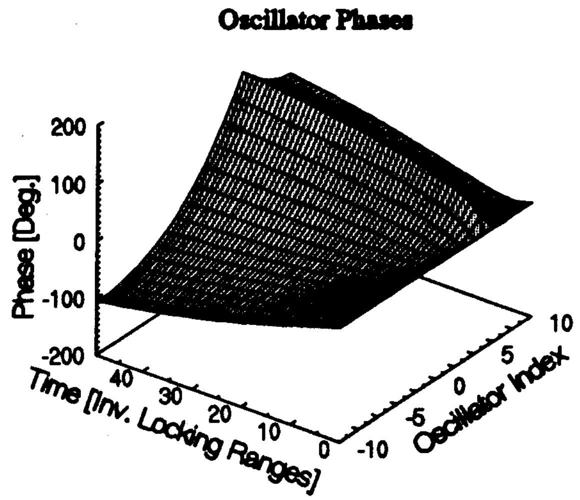

Figure 5.

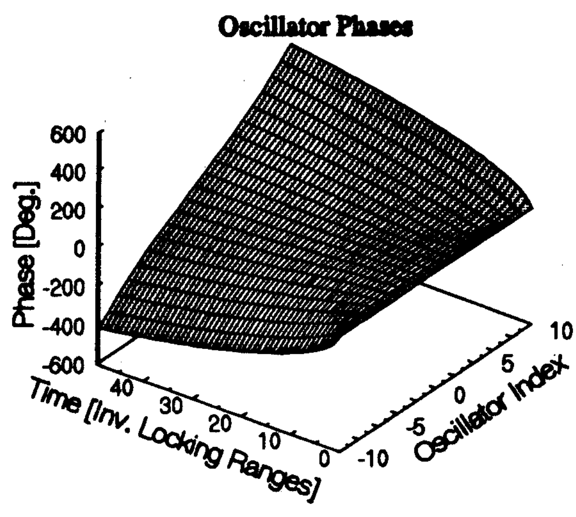

Figure 6

Scanning Beam

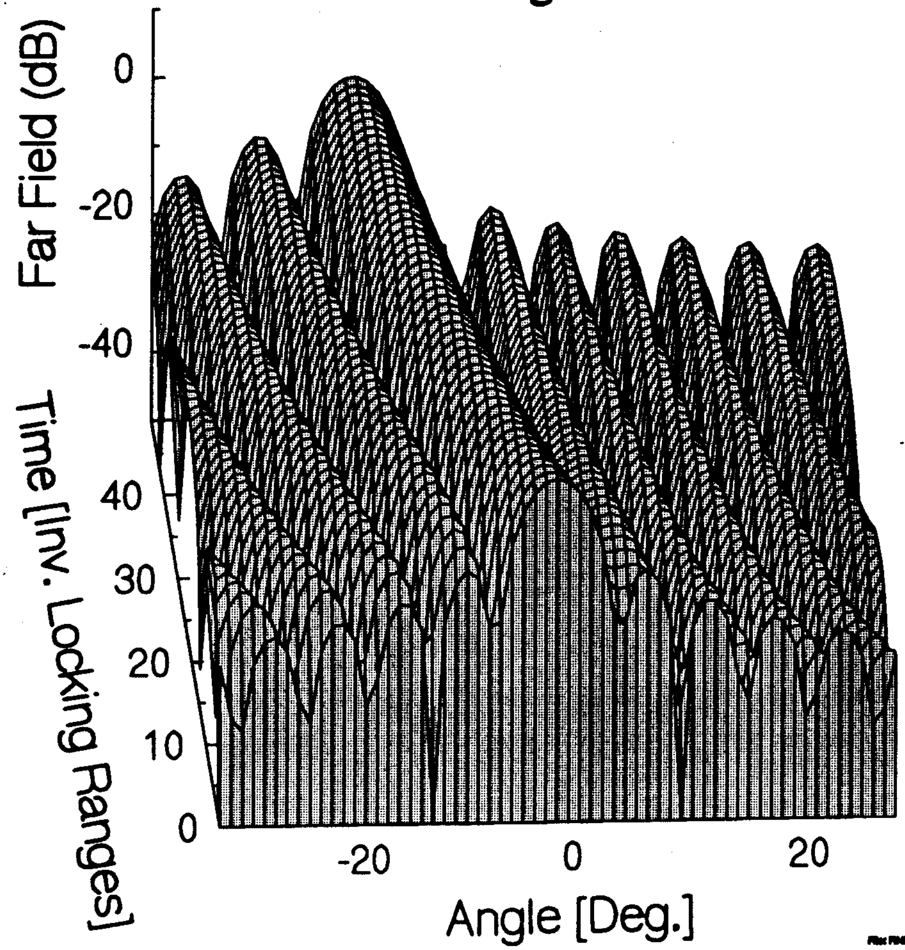

Figure 7.