

Quantum Resonance for Solving NP-Complete Problems By Simulations

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Abstract. Quantum analog computing is based upon similarity between mathematical formalism of a quantum phenomenon and phenomena to be analyzed. In this paper, the mathematical formalism of quantum resonance combined with tensor product decomposability of unitary evolutions is mapped onto a class of NP-complete combinatorial problems.

1. Introduction

The competition between digital and analog computers, i.e., between computations and simulations, has a long history. During the last fifty years the theory of computation has been based implicitly upon classical physics as idealized in the deterministic Turing machine model. However, despite the many successes of digital computers, the existence of so called hard problems has revealed limitations on their capabilities, since the computational time for solving such problems grows exponentially with the size of the problem.

It was well understood that one possible way to fight the "curse" of the combinatorial explosion is to enrich digital computers with analog devices. In contradistinction to a digital computer, which performs operations on numbers symbolizing an underlying physical process, an analog computer processes information by exploiting physical phenomena directly. It is this problem solving via direct simulation that allows an analog approach to reduce the complexity of the computations significantly. This idea was stressed by Feynman [1] who demonstrated that the problem of exponential complexity in terms of calculated probabilities can be reduced to a problem of polynomial complexity in terms of simulated probabilities. Conceptually, a similar approach can be applied to the whole class of NP-complete problems. But is it possible, in general, to find a new mathematical formulation for any intractable problem in such a way that it becomes tractable? Some experts in computational complexity believe that, in the spirit of the Godel theorem, there always exist computational problems such that every mathematical formulation that captures the essence of the problem is intractable [2]. At this step, we cannot prove or disprove this statement.

There are remarkably few (actually three) papers in which quantum analog computing is discussed. The first one [3] introduces a hypothetical quantum device (a slot machine) for solving a traveling salesman problem. As shown by the author, such a device, although intellectually appealing, requires an exponentially large number of measurements to get the right answer. The second paper [4] discusses the capacity of a hypothetical quantum perception. In the third paper [5], a concept of quantum recurrent networks combining quantum conventional networks with classical feedback loops was introduced and discussed.

In this paper an attempt is made to exploit combinatorial properties of tensor product decomposability of unitary evolution of many-particle quantum systems for simulating solutions to NP-complete problems, while the reinforcement and selection of a desired solution is executed by quantum resonance.

2. Quantum Resonance.

Consider a quantum system characterized by a discrete spectrum of energy eigenstates subject to a small perturbing interaction, and let the perturbation be switched on at zero time. The Hamiltonian of the system can be presented as a sum of the time-independent and oscillating components:

$$H = H_0 + \varepsilon_0 H_1 \int_{\omega} \xi(\omega) \sin \omega t d\omega \quad \varepsilon_0 \ll 1 \quad (1)$$

where H_0 and H_1 are constant Hermitian matrices, ω is the frequency of perturbations, and $\varepsilon(\omega)$ is the spectral density.

The probability of a transition from state k to q in the first approximation is proportional to the product [6] :

$$P_{kq} \propto \left| \varphi_k^* H_1 \varphi_q \right|^2 \left[\frac{\sin \frac{1}{2} (a_{qk} - \omega) t}{a_{qk} - \omega} \right]^2 \quad (2)$$

Here φ_j are the eigenstates of H_0 :

$$H_0 \varphi_j = E_j \varphi_j \quad j = 1, 2, \dots, N \quad (3)$$

where E_j are the energy eigenvalues,

$$\hbar a_{kq} = E_k - E_q, \quad k, q = 1, 2, \dots, N \quad (4)$$

and \hbar is the Planck constant.

The resonance, i.e., a time-proportional growth of the transition probability, P_{kq} occurs when $\omega = a_{qk}$:

$$P_{\omega} = \frac{\pi \mathcal{E}_0^2}{\hbar^2} |\varphi_k^* H_1 \varphi_v|^2 \xi^2(\omega) t \quad (5)$$

3. Combinatorial Problems

Combinatorial problems are among the hardest in the theory of computations. They include a special class of so called NP-complete problems that are considered to be intractable by most theoretical computer scientists. A typical representative of this class is a famous traveling-salesman problem (TSP) of determining the shortest closed tour that connects a given set of n points in the plane. As for any of NP-complete problem, here the algorithm for solution is very simple: enumerate all the tours, compute their lengths, and select the shortest one. However, the number of tours is proportional to $n!$ and this leads to exponential growth of computational time as a function of the dimensionality n of the problem, and therefore, to computational intractability.

It should be noticed that, in contradistinction to continuous optimization problems where the knowledge about the length of a trajectory is transferred to the neighboring trajectories through the gradient, here the gradient does not exist, and there is no alternative to a simple enumeration of tours.

The class of NP-complete problems has a very interesting property: if any single problem (including its worse case) can be solved in polynomial time, then every NP-complete problem can be solved in polynomial time as well. But despite that, there is no progress so far in removing a curse of combinatorial explosion: it turns out that if one manages to achieve a polynomial time of computation, then the space or energy grow exponentially, i.e., the effect of combinatorial explosion stubbornly reappears. That is why the intractability of NP-complete problems is being observed as a fundamental principle of theory of computations which plays the same role as the second law of thermodynamics in physics.

At the same time, one has to recognize that the theory of computational complexity is an attribute of a digital approach to computations, which means that the monster of NP-completeness is a creature of the Turing machine. As an alternative, one can turn to an analog device that replaces digital computations by physical simulations. Indeed, assume that one found such a physical phenomenon whose mathematical description is equivalent to that of a particular NP-complete problem. Then, incorporating this phenomenon into an appropriate analog device one can simulate the corresponding NP-complete problem. In this connection it is interesting to note that, at first sight, NP-complete problems are fundamentally different from natural phenomena: they look like man-made puzzles and their formal mathematical framework is mapped into decision problems with yes/no solutions. However, one should recall that physical laws can also be stated in a "man-made" form: The least time (Fermat), the least action (in modifications of Hamilton, Lagrange, or Jacobi), and the least constraints (Gauss).

In this paper we will describe how to map a combinatorial decision problem into the physical phenomenon of quantum resonance on a conceptual level, without going into details of actual implementations.

Let us turn to the property (5) which can be mapped into several computational problems, and, for the purpose of illustration, choose the following one: given n different items to be distributed over n places; the cost of an β^{th} item put in a γ^{th} place is $\lambda_{\beta}^{(\gamma)}$; in general, the costs can be positive or negative, and there are no restrictions to how many different items can be put at the same place. Find yes/no answer to the following question: is there at least one total cost whose absolute value falls into an arbitrarily given interval.

This problem is typical for optimal design. Since the cost of a particular distribution is expressed by the sum

$$E_j = \sum_{\beta=1}^n \lambda_{\beta}^{(\gamma)}, \quad j = 1, 2, \dots, N = n^n \quad (6)$$

classically one has to compute all the n^n sums (8) in order to find is there at least one E_q such that

$$a_1 \leq |E_q| \leq a_2, a_2 > a_1 \quad (7)$$

where a_1 and a_2 are arbitrarily prescribed positive numbers.

Since costs $\lambda_{\beta}^{(\gamma)}$ can be positive or negative, the absolute value in Eq. (7) represents a global constraint, and therefore our problem belongs to the class of so called constraint satisfaction problems that are the hardest among other optimization problems. The constraint (7) prevents one from decomposing the solution into smaller-size sub-problems. As shown by Andre Stechert ^[7], this problem can be mapped into the partition problem ^[8], and therefore, it is NP-complete.

Now we will demonstrate how this problem can be solved by the quantum device described above in one computational step.

First, let us represent the unitary matrix U_0 corresponding to the time-independent Hamiltonian

$$U_0 = e^{iH_0 t} \quad (8)$$

as a tensor product of n diagonal unitary matrices of the size $n \times n$:

$$U_0 = U_1 \otimes U_2 \otimes \dots \otimes U_n \quad (9)$$

where

$$U_\gamma = \begin{pmatrix} e^{i\lambda_1^{(\gamma)}} & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & e^{i\lambda_n^{(\gamma)}} \end{pmatrix} \quad (10)$$

Then the unitary matrix U_0 in (9) will be also diagonal and

$$H_0 = \begin{pmatrix} E_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & E_N \end{pmatrix}, \quad N = n^n \quad (11)$$

while E_j is expressed by Eq. (6).

Hence, if one select $\lambda_j^{(\gamma)}$ in (8) as the costs of a β^{th} item put in a γ^{th} place, then the eigenstates E_j of the Hamiltonian H_0 will represent costs of all $N = n^n$ possible distributions (8).

Now we have to choose the perturbation of the Hamiltonian, (see Eq. (1)) For that purpose assume that initially the quantum device is in a certain base state k , whose energy E_k does not belong to the interval (7), i.e.,

$$|E_k| < a_1, \quad \text{or} \quad |E_k| > a_2 \quad (12)$$

and select H_1 and $\xi(\omega)$ as follows:

$$H_1 = P \quad (13)$$

where

$$P = \begin{pmatrix} 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 1 \end{pmatrix} \quad (14)$$

and

$$\xi(\omega) = \begin{cases} \xi_0 = \text{constant} & \text{if } \frac{|E_k - a_2|}{\hbar} \leq \omega < \frac{|E_k - a_1|}{\hbar} \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

Here, for the sake of concreteness, the initial state E_k was selected such that:

$$|E_k - a_1| > |E_k - a_2| \quad (16)$$

Suppose that the given interval a_1, a_2 contains at least one total cost $|E_q|$ from the set (6), i.e., $|E_q|$ satisfies the inequality (7). Then, according to Eqs. (5) and (14), the resonance transition from the initial state E_k to the state E_q (or other states satisfying (7)) will occur with the probability one. Indeed, in the presence of a resonance, the probability for non-resonance transitions are vanishingly small if $\epsilon_0 \ll 1$ (see Eq. (1)).

However, if the given interval a_1, a_2 does not contain any costs $|E_q|$ from the set (6), then according to Eqs. (5) and (14), there will be no resonance transitions at all, and therefore, with the probability one the quantum device will stay in the initial state.

Thus, in one computational step, the problem is solved in a deterministic way. As follows from Eq. (5), the time required for probability of the resonance transition from the state k to q to become close to one has the order:

$$t^* \sim O\left(\frac{\hbar^2}{\varepsilon_0^2 \xi^2 |\varphi_k^* H_1 \varphi_q|^2}\right) \quad (17)$$

4. Conclusion

Thus, it has been demonstrated how a "man-made" problems of exponential computational complexity which is hard to handle by algorithmic methods are solved by exploiting a strongly pronounced physical phenomena: quantum resonance.

The main advantage of the proposed approach is in exponential speedup of solutions to NP-complete combinatorial problems. Two fundamental physical phenomena contribute to it: quantum resonance and tensor-product decomposability of the underlying unitary matrix.

Quantum resonance allows one to represent all the possible solutions to the problem as a set of competing dynamical processes: energy exchanges between pairs of quantum eigenstates. The mathematical formalism of quantum resonance provides a storage for these processes: the transition matrix P_{kq} (see Eq. (2)) where each process is labeled through the corresponding transition probability.

Tensor-product decomposability is a fundamental property of the Schrodinger equation for multi-particle systems. Due to its effect, the number of stored solutions, i.e., the number of transitions P_{kq} is exponentially larger than the number of the input parameters (see Eq. (6)) and that is what directly contributes into exponential speedup and capacity.

In order to make these two physical phenomena work together, one has to choose the Hamiltonian of the quantum system such that the optimal solution is the winner in the competition with other solutions, i.e., that its transition probability is the largest. This is achieved by selecting the oscillating part of the Hamiltonian in the form of (14).

It should be emphasized that the solution of one NP-complete problem opens up a way to solve every NP-complete problem in polynomial time.

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