# Serial Concatenated Trellis Coded Modulation with Rate-1 Inner Code

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## Abstract

We develop new, low complexity turbo codes suitable for bandwidth and power limited systems, for very low bit and word error rate requirements. Motivated by the structure of recently discovered low complexity codes such as Repeat-Accumulate (RA) codes with low density parity check matrix, we extend the structure to high-level modulations such as 8PSK, and 16QAM. The structure consists of a simple 4-state convolutional or short block code as an outer code, and a rate-1, 2 or 4-state inner code. Two design criteria are proposed: the maximum likelihood design criterion, for short to moderate block sizes, and an iterative decoding design criterion for very long block sizes.

## 1 Introduction

Trellis coded modulation (TCM) proposed by Ungerboeck in 1982 [1] is now a well-established technique in digital communications. Since its first appearance, TCM has generated a continuously growing interest, concerning its theoretical foundations as well as its numerous applications, spanning high-rate digital transmission over voice circuits, digital microwave radio relay links, and satellite communications. In essence, it is a technique to obtain significant coding gains (3-6 dB) sacrificing neither data rate nor bandwidth.

Turbo codes represent a more recent development in the coding research field [2], which has raised a large interest in the coding community. They are *parallel concatenated convolutional codes* (PCCC) whose encoder is formed by two (or more) *constituent* systematic encoders joined through one or more interleavers.

The suboptimal iterative decoding structure is modular, and consists of a set of concatenated decoding modules, one for each constituent code, connected through the same interleaver used at the encoder side. Each decoder performs weighted soft decoding of the input sequence. Parallel concatenated convolutional codes yield very large coding gains at the expense of a data rate reduction, or bandwidth increase. In [4] we merged TCM and PCCC in order to obtain large coding gains and high bandwidth efficiency.

For certain applications, we require very low bit error rates ($10^{-9}$). To achieve this goal we have suggested to merge TCM with the recently discovered serial concatenated codes (SCCC) [5], which have lower error floors, and adapting the concept of iterative decoding used in parallel concatenated codes. We note that optimizing the overall code with large deterministic interleavers was prohibitively complex. However, by using the concept of *uniform interleaver* [6] it is possible to draw some criteria to optimize the component codes for the construction of powerful serial concatenated codes with large block size. The optimum decoding of such complex codes is practically impossible. Only the use of suboptimum iterative decoding methods makes it possible to decode such complex codes. In the following, we will call the concatenation of an outer convolutional or a short block code with an inner TCM a *serially concatenated TCM* (SCTCM). For parallel concatenated trellis coded modulation (PCTCM), also addressed as "turbo TCM", a first attempt employing the so-called "pragmatic" approach to TCM was described in [7]. Later, turbo codes were embedded in multilevel codes with multistage decoding [8]. Recently, punctured versions of Ungerboeck codes were used to construct turbo codes for 8PSK modulation [9]. In [4] we proposed a new solution to PCTCM with multilevel amplitude/phase modulations, and a suitable bit-by-bit iterative decoding structure. Preliminary results [4] showed that the performance of the proposed codes is within 1 dB from the Shannon limit at bit error probabilities of $10^{-7}$ for large block sizes. Unfortunately PCTCM [4] and all other proposed schemes in [7], [8] and [9] may produce an error floor above $10^{-9}$. However, SCTCM is expected to have much lower error floor, based on maximum likelihood (ML) analysis. Two design criteria for serially concatenated trellis coded modulation (SCTCM) are proposed. The ML design criterion, for short to moderate block sizes, and an iterative turbo decoding design criterion for very long block sizes.

## 2 SCTCM: Code Design

The basic structure of serially concatenated trellis coded modulation was proposed in [13] and is shown in Fig. 1. We developed a method to design serial concatenated TCM, which achieves $b$ bits/sec/Hz, using a rate $2b/(2b+1)$ binary convolutional encoder (or a short block code) with maximum free Hamming distance (or minimum distance) as the outer code. An interleaver $\pi$ permutes the output of the outer code. The interleaved

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data enters a rate $(2b+1)/(2b+2)$ recursive convolutional inner encoder. The $2b+2$ output bits are then mapped to two symbols each belonging to a $2^{b+1}$ level modulation (four dimensional modulation). In this way, we are using $2b$ information bits for every two modulation symbol intervals, resulting in $b$ bit/sec/Hz transmission (when ideal Nyquist pulse shaping is used) or, in other words, $b$ bits per modulation symbol. The inner code and the mapping are jointly optimized based on maximizing the effective free Euclidean distance of the inner TCM.

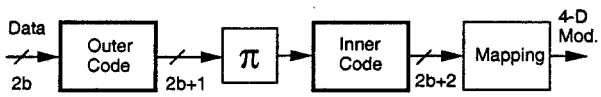


Figure 1: Block Diagram of the Encoder for SCTCM.

## 2.1 Rationale for Low-Complexity Code Selection

Although the above SCTCM structure results in a powerful code, we note that the number of transitions per state for the inner TCM is $2^{2b+1}$. For the case of interest we have $b=3$. Thus even if we keep the number of states low say 2, we have 128 transitions per state, which results in 256 edges in the trellis section. The complexity of the decoder depends on the number of edges per trellis section. Therefore for high speed operation we can't afford such complexity.

Our recent results on concatenation of an outer code with a simple accumulator as inner code for binary modulation [16] [14] led us to develop a second method for serial concatenated TCM (SCTCM). For MPSK, or a two dimensional constellation with $M$ points, let's define $m=\log_2 M$, where M is number of phases. We propose a novel method, with lower complexity, to design serial concatenated TCM, which achieves $bm/(b+1)$ bits/sec/Hz, using a rate $b/(b+1)$ binary convolutional encoder (or a short block code) with maximum free Hamming distance (or minimum distance) as the outer code. An interleaver $\pi$ permutes the output of the outer code. The interleaved data enters a rate $m/m=1$ recursive convolutional inner encoder. The $m$ output bits are then mapped to one symbol belonging to a $2^m$ level modulation. The structure of the SCTCM encoder is shown in Fig. 2.

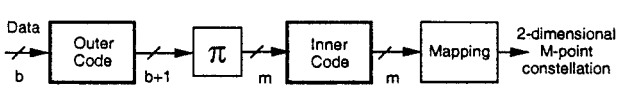


Figure 2: Structure of the encoder for serial concatenated trellis coded modulation. (2-D, M-point constellation).

In this way, we are using $b$ information bits per $\frac{b+1}{m}$ modulation symbol interval, resulting in $bm/(b+1)$ bit/sec/Hz transmission (when ideal Nyquist pulse shaping is used) or, in other words, $bm/(b+1)$ bits per modulation symbol. The inner code and the mapping are jointly optimized based on maximizing the effective free Euclidean distance of the inner TCM. For example consider 8PSK modulation, where $m=3$, then the throughput $r=3b/(b+1)$ is as follows: for $b=2$, $r=2$; for $b=3$, $r=2.25$; and for $b=4$, $r=2.4$. This suggest that we can use a rate 1/2 convolutional code with puncturing to obtain various throughputs without changing the inner code or modulation.

For rectangular $M^2$-QAM, where $m=\log_2 M$, the structure becomes even simpler. In this case, to achieve throughput of $2mb/(b+1)$ bps/Hz we need a rate $b/(b+1)$ outer code and a rate $m/m$ inner code, where the $m$ output bits are alternatively assigned to in-phase and quadrature components of the $M^2$-QAM modulation. The structure of the SCTCM encoder is shown in Fig. 3. For example consider 16-QAM modulation, where $m=2$, then the throughput $r=4b/(b+1)$ is: for $b=1, r=2$; for $b=2, r=2.67$; for $b=3, r=3$; and for $b=4, r=3.2$.

For the case of interest we have $b=r=3$. We note that now the number of transitions per state of the inner TCM is reduced to 4 (this results in a large reduction in complexity: 32 times lower than the previous case). Moreover, the outer code also has lower code rate (from 6/7 to 3/4). Here we only consider the example of 16QAM modulation, and $r=3$ which implies $b=3$. The encoder structure of SCTCM for 2-state inner TCM is shown in Fig. 4. The encoder structure of SCTCM for 4-state inner TCM is shown in Fig. 5.

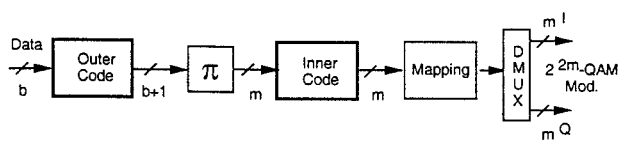


Figure 3: Structure of the encoder for serial concatenated trellis coded modulation. ($M^2$ QAM).

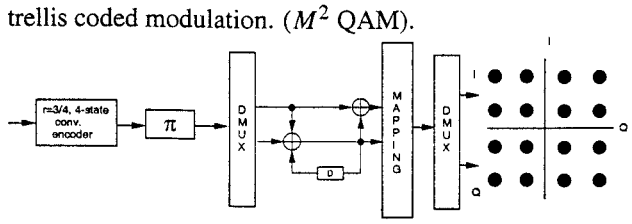


Figure 4: 3 bps/Hz Turbo Trellis Coded Modulation with 2-state inner TCM

The output of inner encoder in each case is mapped to the I and Q components of 16QAM alternatively . The outer code is an optimum rate 3/4, 4-state nonrecursive convolutional code with free Hamming distance of 3. The structure of outer encoder is shown in Fig. 6.

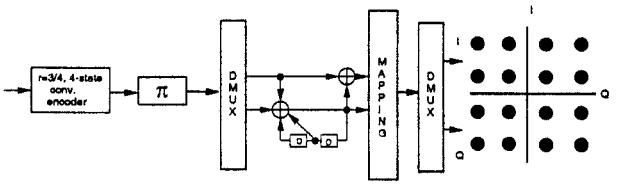


Figure 5: 3 bps/Hz Turbo Trellis Coded Modulation with 4-state inner TCM

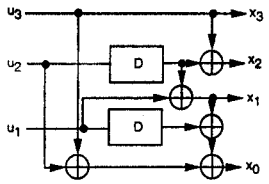


Figure 6: Optimum rate 3/4, 4-state outer code

The optimum rate 3/4, 4-state outer code has 32 edges per trellis section and produces 4 output bits. Thus the complexity per output bit is 32/4=8. The complexity per input bit is 32/3. To further reduce the complexity of the outer code we suggest to use a rate 1/2, 4-state systematic recursive convolutional code. This code can be punctured to rate 3/4, by puncturing only the parity bits. The minimum distance of this punctured code is 3, the same as for optimum code. Now the code has 8 edges per trellis section and produces 2 output bits. Thus the complexity per output bit is 8/2=4. Since this code is systematic there is no complexity associated with the input bit (see description of SISO for outer code in iterative decoding). The encoder structure for this low complexity SCTCM is shown in Figure 7. If we use the proposed low complexity SCTCM with 4-state outer and 4-state inner, with no correction terms in the SISO module, then the complexity of the proposed scheme with 5 iterations will be roughly equal to the complexity of a standard Viterbi decoder, but still obtaining a 2 dB advantage over the Pragmatic TCM system to be discussed later.

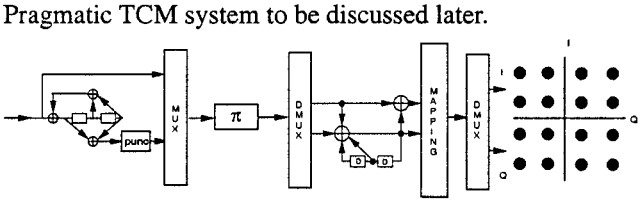


Figure 7: 3 bps/Hz low-complexity serial turbo trellis coded modulation

## 2.2 Maximum Likelihood Decoding Design Criteria for SCTCM

It can be shown that the dominant term in the transfer function bound on bit error probability of serially concatenated TCM, employing an outer code with free (or minimum) Hamming distance $d_f^o$, averaged over all possible interleavers of size $N$ bits, is proportional for large $N$ to

$$N^{-\lfloor (d_f^o+1)/2 \rfloor} e^{-\delta^2 (E_s/4N_o)}$$

where $\lfloor x \rfloor$ represents the integer part of $x$, and

$$\delta^2 = \frac{d_f^o d_{\mathrm{f,eff}}^2}{2}, \quad \text{for } d_f^o \text{ even, and}$$

$$\delta^2 = \frac{(d_f^o - 3) d_{\mathrm{f,eff}}^2}{2} + (h_m^{(3)})^2, \quad \text{for } d_f^o \text{ odd.}$$

The parameter $d_{\mathrm{f,eff}}$ is the *effective free Euclidean distance* of the inner code (to be defined in the following), $h_m^{(3)}$ is the minimum Euclidean distance of inner code sequences generated by input sequences with Hamming distance 3, and $E_s/N_o$ is the $M$-ary symbol signal-to-noise ratio.

The above results are valid for very large $N$.

Based on these results, the design criterion for serially concatenated TCM for large interleavers and very low bit error rates is to maximize the free Hamming distance of the outer code (to achieve interleaving gain), and to maximize the effective free Euclidean distance of the inner TCM code.

Let $\mathbf{z}$ be the binary input sequence to the inner TCM code, and $x(\mathbf{z})$ be the corresponding inner TCM encoder output with $M$-ary symbols. The criteria proposed for designing and selecting the constituent inner TCM encoder are the following:

1. Design the constituent inner TCM encoder for a given two or multidimensional modulation such that the minimum Euclidean distance $d(x(\mathbf{z}), x(\mathbf{z}'))$ over all $\mathbf{z}, \mathbf{z}'$ pairs, $\mathbf{z} \neq \mathbf{z}'$, is maximized, given that the Hamming distance $d_H(\mathbf{z}, \mathbf{z}') = 2$. We call this minimum Euclidean distance the *effective free Euclidean distance* of the inner TCM code and denote it simply by $d_{\mathrm{f,eff}}$.

2. If the free distance of the outer code $d_f^o$ is odd, then, among the selected inner TCM encoders, choose those that have the maximum Euclidean distance $d(x(\mathbf{z}), x(\mathbf{z}'))$ over all $\mathbf{z}, \mathbf{z}'$ pairs, $\mathbf{z} \neq \mathbf{z}'$, given that the Hamming distance $d_H(\mathbf{z}, \mathbf{z}') = 3$. We call this the minimum Euclidean distance of the inner TCM code due to input Hamming distance 3, and denote it by $h_m^{(3)}$.

## 2.3 Mapping (output labels) for TCM

As soon as the input labels and output signals are assigned to the edges of a trellis we have a complete description of the TCM code. The selection of the mapping (output labels) does not change the trellis code. However, it influences the encoder circuit required to implement the TCM scheme. A convenient mapping should be selected to simplify the encoder circuit and, if possible, to

yield a linear circuit that can be implemented with exclusive ORs. The set partitioning of the constellation and the assignment of constellation points to trellis edges, and the successive assignments of input labels to the edges are important. Ungerboeck [1] proposed a mapping called *mapping by set partitioning*, leading to the "natural mapping". This mapping for two-dimensional modulation is useful if one selects the TCM scheme by searching among all encoder circuits that maximize the minimum Euclidean distance.

## 2.4 Design Method for Inner TCM

The proposed design method is based on the following steps:

1. The well known set partitioning techniques for signal sets are used (see for example [10] and the references therein).

2. The input labels assignment is based on the codewords of the parity check code $(m, m-1, 2)$ and its set partitioning, to maximize the quantities described in subsection 2.2. Using this method the minimum Hamming distance between input labels for parallel transitions will be equal to 2. The assignment of codewords of the parity check code as input labels to the 2-dimensional signal points is not arbitrary.

3. A sufficient condition to have very large output Euclidean distances for input sequences with Hamming distance 1, is that all input labels to each state be distinct.

4. Assign pair of input labels and 2-dimensional signal points to the edges of a trellis diagram based on the design criteria in subsection 2.2.

## 2.5 Example of the Design Methodology

**Example 1: Set partitioning of 8PSK and input labels assignment.**

Let the eight phases of 8PSK be denoted by $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Here $m = 3$. Consider the 8PSK signal set $A = [0, 2, 4, 6]$, and set $B = [1, 3, 5, 7]$. For unit radius 8PSK constellation, the minimum intra-set square Euclidean distance for each set is 2. The minimum inter-set square Euclidean distances 0.586 .

Select the input label set $L_0$ as codewords of the $(3, 2, 2)$ parity check code, i.e. $L_0 = [(000), (011), (101), (110)]$, next generate input label $L_1 = L_0 + (001)$, i.e., $L_1 = [(001), (010), (100), (111)]$. Consider a 2-state trellis. Assign the input-output pair $(L_0, A)$ to four edges from state 0 to state 0. Assign the input-output pair $(L_1, B)$ to four edges from state 0 to state 1. Next assign the input-output pair $(L_2, A)$ to four edges from state 1 to state 0, and assign the input-output pair $(L_3, B)$ to four edges from state 1 to state 1. $L_2$ has the same elements as in $L_1$ but with different order, and $L_3$ has the same elements as in $L_0$ again with different order. In order to maximize the minimum Euclidean distance due to the input sequences with Hamming distance 2, we have to find the right permutation within each set. In this case it turns out that using the complement operation suffices. Therefore define input label $L_2$ as the complement of the elements of $L_1$ without changing their order, i.e., $L_2 = [(111), (100), (010), (001)]$. Finally $L_3$ is generated in the same way, as the complement of the elements in $L_0$, i.e., $L_3 = [(110), (101), (011), (000)]$.

Such assignment guarantees that the squared effective free Euclidean distance of trellis code is 2, where the minimum squared Euclidean distance of the code is 0.586.

Having determined the code by its input labels and 2-dimensional output signals, the encoder structure can then be obtained by selecting any appropriate labels (output labels) for the 2-dimensional output signals. We used the following output mapping, $[(000), (001), (010), (011), (110), (111), (100), (101)]$, mapped to phases $[0, 1, 2, 3, 4, 5, 6, 7]$, which is called "reordered mapping". The Truth Table required to implement the code is shown in Table 1. The implementation of the 2-state inner trellis code was obtained from the Truth Table as shown in Fig. 8. For this 2-state inner code, $d^2_{\text{f,eff}} = 2$, $h^{(3)}_m = \infty$, and $h^2_m = 0.586$. The outer code for this example can been selected as an 4-state, rate 2/3, convolutional code with $d^o_f = 3$ (this is a recursive systematic rate 1/2 convolutional code where the parity bits are punctured). Since $h^{(3)}_m = \infty$ then $d^o_f$ is increased effectively to 4. This method of design is appropriate for short to moderate block sizes. It results in having low error floors, thus achieving low bit and frame error rates.

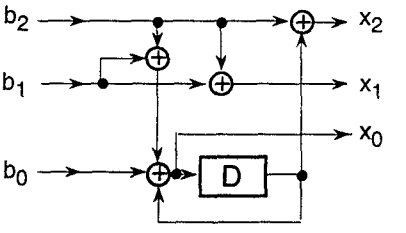


Figure 8: Optimum 2-state inner trellis encoder for SCTCM with 8PSK modulation (Optimized using ML criterion).

| P-State | Input Label | N-State | Phases | Output Label |
|---|---|---|---|---|
| $p$ | $b_2, b_1, b_0$ | $n$ | $\phi$ | $x_2, x_1, x_0$ |
| 0 | 000 | 0 | 0 | 000 |
| 0 | 011 | 0 | 2 | 010 |
| 0 | 101 | 0 | 4 | 110 |
| 0 | 110 | 0 | 6 | 100 |
| 0 | 011 | 1 | 1 | 001 |
| 0 | 010 | 1 | 3 | 011 |
| 0 | 100 | 1 | 5 | 111 |
| 0 | 111 | 1 | 7 | 101 |
| 1 | 111 | 0 | 0 | 000 |
| 1 | 100 | 0 | 2 | 010 |
| 1 | 010 | 0 | 4 | 110 |
| 1 | 001 | 0 | 6 | 100 |
| 1 | 110 | 1 | 1 | 001 |
| 1 | 101 | 1 | 3 | 011 |
| 1 | 011 | 1 | 5 | 111 |
| 1 | 000 | 1 | 7 | 101 |

Table 1: Truth Table for the 2-state inner TCM with 8PSK.

## 2.6 Iterative Decoding Design Criteria for SCTCM

The design criterion is based on the method of density evolution proposed by Richardson and Urbanke [18]. Using this method, they computed the capacity threshold for Low density parity check codes over binary input AWGN channels. Wiberg [17] in his dissertation has shown that the extrinsic information can be approximated by Gaussian density function. Hagenauer [], and El Gamal [19] considered the soft-input soft-output APP module as a signal-to-noise ratio (SNR) transformer. Using these ideas, and El Gamal's [19] suggested method to analyze turbo codes, we extended the results to analyze concatenated TCM by approximating the density functions for extrinsics as Gaussian densities. Several methods now can be used, under Gaussian assumption, the more accurate results can be obtained by using the method of Richardson and Urbanke. Less accurate result but with easier computation can be obtained if we use density consistency suggested by Richardson, and Urbanke. At each iteration, we computed SNRs using random data (code is nonlinear and its performance requires to be averaged over the transmitted patterns), and collected them for the outer and the inner codes. We used the example of 4-state outer with puncturing pattern 100100... and 4-state rate-1 inner as shown in Figure 7. The input-output SNR for inner and outer codes are shown in Figure 9. If the two curves do not cross, then the iterative decoder converges. Note that we used all assumptions made by Richardson and Urbanke for very large block sizes (essentially when the block size and the number of iterations go to infinity but the number of iterations are much less than roughly log of the block size corresponding to the girth of the graph representing the overall code). In Figure 9 we see that if $Eb/No$ is greater than 4.8 dB, the iterative decoder converges, where the capacity limit is 4.54 dB, so this code never reaches the capacity limit no matter how large is the block size and the number of iterations. This method was used to select the 2-state and the 4-state inner TCM codes.

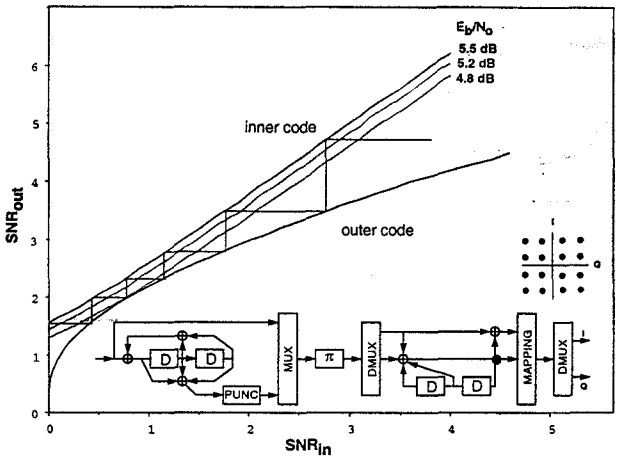


Figure 9: Iterative Decoding threshold computation for SCTCM with punctured pattern 100, 4-state outer and rate-1, 4-state inner for 16QAM, throughput of 3 bits per 16QAM symbol.

We have also obtained thresholds for 8PSK modulation and selected the optimum code under the iterative decoding design criterion, as shown in Fig. 10.

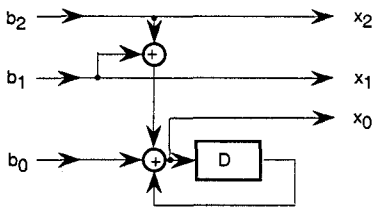


Figure 10: 2-state inner trellis encoder for SCTCM with 8PSK modulation, optimized using iterative decoding criterion.

## 2.7 Simulation Results for serial TCM with rate 1 inner code

The iterative decoder performance of the proposed low complexity turbo serial TCM is shown in Fig. 11. In the simulations we used an optimum rate 3/4 outer code as shown in Fig. 6. Simulations for the punctured low complexity turbo serial TCM as shown in Fig. 7, for 2-state and 4-state inner TCM, are in progress.

The capacity of this signal set at throughput 3 bps/Hz, is $E_b/N_o$=4.54 dB, and at throughput 3.28 bps/Hz is

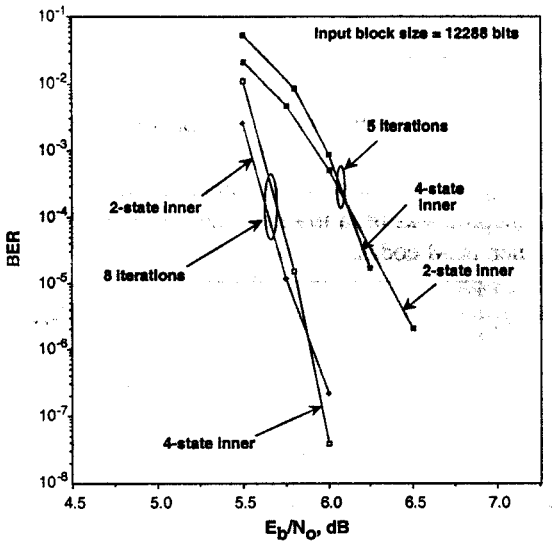


Figure 11: Performance of 3 bps/Hz Turbo Trellis Coded Modulation

$E_b/N_o$=5.36 dB. The Pragmatic TCM system designed for throughput 3.28 bps/Hz, at BER= $10^{-9}$ requires about 9.5 dB. Therefore the Pragmatic TCM system is 4.15 dB away from the capacity limit. The structure of iterative decoder for punctured outer code is shown in Fig. 12. Description of SISO can be found in [11], and [12].

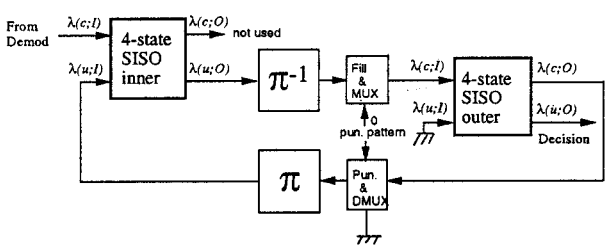


Figure 12: Turbo decoder low complexity SCTCM

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