On the Validity of the Double Integrator Approximation in Deep Space Formation Flying

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1. Introduction

Two formations flying missions, Terrestrial Planet Finder (TPF) and its precursor, Starlight, are baselined for Earth-trailing, deep space orbits. In deep space it is often assumed during development that the formation is "free-flying," that is, the relative translational dynamics between members of the formation may be approximated as double integrators [4, 6].

Starlight requires centimeter level accuracy and TPF requires millimeter level accuracy, both missions operating with up to a kilometer of spacecraft separation [1, 3]. So, on the one hand, the dynamics are simplified to double integrators for simulation and design, but on the other hand, the simulations must be very accurate to determine whether or not the performance requirements have been satisfied.

Paraphrasing Simmonds and Mann, small effects acting over long periods can have large effects. Numerous small effects are ignored in the free-flying assumption, and, over sufficiently small time scales, these effects can justifiably be ignored. The question that has not been quantitatively answered previously is: over what time scale are these free-flying models valid?

This paper first analyzes the relative formation dynamics to precisely quantify the numerous small effects. Given this quantification, the main contribution of the paper is analytic expressions that upper bound the error between free-flying models and the full, nonlinear dynamics. Given a formation flying mission scenario, the bounds can be used to quickly evaluate the needed modeling fidelity. Also, these bounds are shown to be reasonably tight using a preliminary TPF mission design.

Numerical simulations introduce inaccuracies due to quantization. In the example at the end of the paper, the numerical integration of the full, nonlinear model was done in MATHEMATICA with 25 digits of accuracy, an accuracy equivalent to $10^{-12}\,m$ at 1 astronomical unit.

2. Formation Translational Dynamics

Figure 1 depicts some of the variables to be used. An inertial frame $\mathcal{F}_I$, with origin O_I, is located at the center of the Sun. The formation frame, $\mathcal{F}_F$, based on an Earth-trailing orbit, has its origin moving in a circular orbit at 1 AU. $\mathcal{F}_F$ has identical axes as $\mathcal{F}_I$: $\mathcal{F}_F$'s origin is translating, but $\mathcal{F}_F$ is not rotating with respect to $\mathcal{F}_I$.

The equation of translational motion for the ith spacecraft, $i = 1, 2, \ldots$, is taken from [2]: $\ddot{\mathbf{r}}_i = -\mu_s \frac{\mathbf{r}_i}{|\mathbf{r}_i|^3} + \mathbf{b}_i + \mathbf{c}_i + \mathbf{u}_i$, where $\mathbf{r}_i$ is the position vector of the spacecraft with respect to O_I, $\mathbf{b}_i$ is the acceleration due to solar pressure, $\mathbf{c}_i$ is the acceleration due to other body effects and $\mathbf{u}_i$ is the control input. Simplifying assumptions are made concerning the solar pressure so that it may be expressed as $\mathbf{b}_i = \beta_i \mathbf{r}_i / |\mathbf{r}_i|^3$ where β_i is referred to as the solar pressure coefficient.

For simplicity of notation, let us assume spacecraft 1 is a "base" body, and define $\boldsymbol{\rho}_i \triangleq \mathbf{r}_{i/1} =$

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$\mathbf{r}_i - \mathbf{r}_1 = \mathbf{r}_{i/F} - \mathbf{r}_{1/F}$. Then, as derived in more detail in the paper:

$$\ddot{\boldsymbol{\rho}}_i = -\frac{\mu_s}{|\mathbf{r}_F|^3}\mathbf{Q}_{\mathbf{r}_F}\boldsymbol{\rho}_i + \left((\beta_i - \beta_1)\frac{\mathbf{r}_F}{|\mathbf{r}_F|^3} + \frac{\beta_i}{|\mathbf{r}_F|^3}\mathbf{Q}_{\mathbf{r}_F}\boldsymbol{\rho}_i + \frac{(\beta_i - \beta_1)}{|\mathbf{r}_F|^3}\mathbf{Q}_{\mathbf{r}_F}\mathbf{r}_{1/F}\right) - \sum_{\text{planets}} \frac{\mu_{\text{planet}}}{|\mathbf{r}_{F/\text{planet}}|^3}\mathbf{Q}_{\mathbf{r}_{F/\text{planet}}}\boldsymbol{\rho}_i$$
$$+ (\mathbf{u}_i - \mathbf{u}_1) + (\mathcal{E}_i - \mathcal{E}_1) \tag{1}$$

where $\mathcal{E}_i$ and $\mathcal{E}_1$ are remainder terms from Taylor expansions, $\mathbf{Q_r} \triangleq (\mathbf{1} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})$, $\mathbf{1}$ is the unit tensor and $\hat{\mathbf{r}}\hat{\mathbf{r}}$ is a direct product with itself of the unit vector associated with $\mathbf{r}$. The solar pressure terms (terms including β's) represent, in order: the relative acceleration of two spacecraft at the same location, $\mathbf{r}_F$, with differing solar pressure coefficients, β_i and β_1; the relative acceleration of two spacecraft of identical solar pressure coefficient, β_i, separated by $\boldsymbol{\rho}_i$; and an offset term to account for the fact that spacecraft 1 is not located at $\mathbf{r}_F$. They are referred to as the Q term, the DC term and the Offset term, respectively. The derivation of (1) is similar to a derivation in Ref. 5, except now solar pressure and third body effects have been included and, most importantly, since the Taylor remainders have been included (1) is *equal to* the full, nonlinear model.

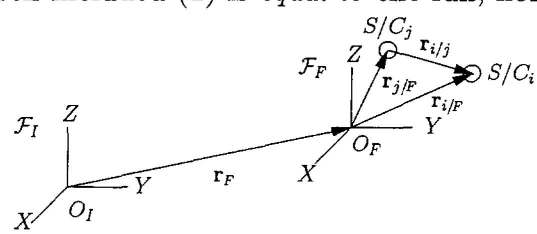

Figure 1: Frames for Formation Dynamics

By assumption $|\mathbf{r}_F| = 1$ AU. Further assume $|\boldsymbol{\rho}_i|$, $|\mathbf{r}_{1/F}| \le 1\ km$, as is the case for TPF and Starlight. Using preliminary Starlight and TPF mission designs, the magnitude of the terms in (1) can be bounded. Shown in the full paper, the result is that the solar pressure DC term dominates all other terms by three orders of magnitude, and the acceleration due to the thrusters is another two orders of magnitude above that. Further, in ranking the terms based on magnitude, they fall into groups, each group separated by two or three orders of magnitudes from the subsequent group.

Given the spread in magnitudes, the common assumption is to approximate the relative dynamics of (1) by $\ddot{\boldsymbol{\rho}}_i^n = (\mathbf{u}_i - \mathbf{u}_1)$, where the superscript "n" represents "no" disturbance, or by $\ddot{\boldsymbol{\rho}}_i^c = (\mathbf{u}_i - \mathbf{u}_1) + \mathbf{d}_i$, where the superscript "c" represents "constant" disturbance and $\mathbf{d}_i = (\beta_i - \beta_1)\mathbf{r}_F(0)/|\mathbf{r}_F(0)|^3$.

3. Error Analysis

Unbolded symbols are representations in $\mathcal{F}_I$ of the associated vectors (bold symbols) of the previous section. The errors between the simplified ρ_i^n and ρ_i^c models and the full nonlinear dynamics are $\lambda_n \triangleq \rho_i^n - \rho_i$ and $\lambda_c \triangleq \rho_i^c - \rho_i$. The process for bounding λ_n is shown in brief. The process is the similar for λ_c, but more involved as more terms are being considered. The dynamics for λ_n are first partioned based on the magnitude groupings of the terms noted in the previous section: $\ddot{\lambda}_n = \mathrm{d}_i + \epsilon^c(t, \rho_i^c)$, where $\epsilon^c(t, \rho_i^c)$ contains all the other terms in (1) except for d_i. For TPF, d_i is approximately $7e - 7\,m/s^2$. Also, a time dependent bound, $\bar{\epsilon}_t^c$, can be derived such that $\|\epsilon^c(\sigma, \rho_i^c)\| \le \bar{\epsilon}_t^c$ for $\sigma \in [0, t]$. Since

$$\|\lambda_n(t)\| = \left\|\int_0^t \int_0^\theta (\mathrm{d}_i + \epsilon^c(\sigma, \rho_i^c))\, \mathrm{d}\sigma\, \mathrm{d}\theta\right\| \le \int_0^t \int_0^\theta \|\mathrm{d}_i\|\, \mathrm{d}\sigma\, \mathrm{d}\theta + \int_0^t \int_0^\theta \bar{\epsilon}_t^c\, \mathrm{d}\sigma\, \mathrm{d}\theta\,,$$

we have $\|\lambda_n(t)\| \le (\|\mathrm{d}_i\| + \epsilon_t^c)t^2/2$.

Assume the model must be accurate to δ, for example, $1\,mm$. To bound the length of time that the "no disturbance" model is accurate to δ, $(\|\mathrm{d}_i\| + \epsilon_t^c)t^2/2$ can be plotted versus time and the value when it equals δ determined. There is also an iterative procedure described in the paper.

For λ_c, two bounds are derived: a simpler bound that is cubic in t and a more complex, but much tighter bound.

4. Numerical Example

Figure 2.a and Figure 2.b show the bounds for $\|\lambda_n(t)\|$ and $\|\lambda_c(t)\|$ and the actual values for $\|\lambda_n(t)\|$ and $\|\lambda_c(t)\|$. The actual values were calculated in MATHEMATICA for an initial condition that is near worst-case in terms of model error. TPF values were used for spacecraft specifics, and a δ of $1\,mm$ is shown in the figures.

The bound for $\|\lambda_n(t)\|$ is indistinguishable from the true model error, as is the complex bound for $\|\lambda_c(t)\|$. The simple bound for $\|\lambda_c(t)\|$ is still relatively tight over the time of interest. The no-disturbance free-flying model is only accurate to $1\,mm$ for 53 seconds, and the constant-disturbance free-flying model is only accurate to $1\,mm$ for 3200 seconds, which is less than an hour. As TPF has a number of 8 hour maneuvers, more complicated models must be used. Future research will consider quantifying the time of validity for further, incremental increases in model complexity—such research would indicate when Hill's equations must be used.

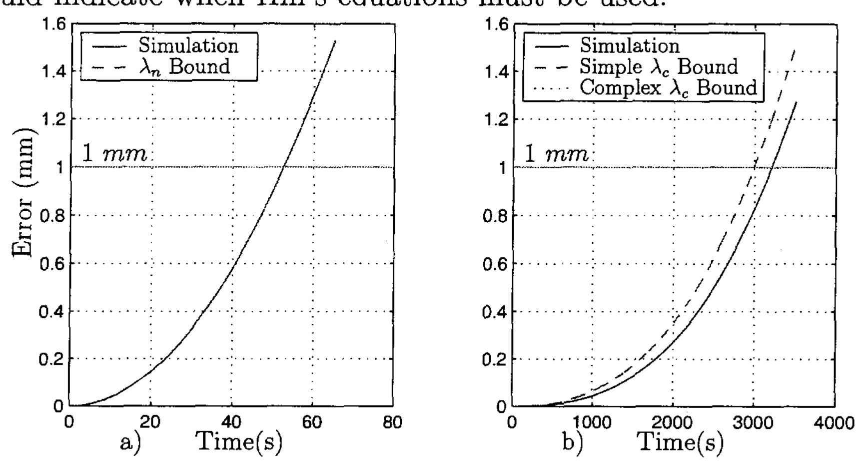

Figure 2: Errors between Full Dynamics and Simplified Models with Bounds

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