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APPLICATION OF TISSERAND'S CRITERION TO THE DESIGN OF GRAVITY ASSIST TRAJECTORIES

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ABSTRACT

Gravity assist involves the use of the gravitational attraction of an intermediate planet to increase the orbit energy of a spacecraft, enabling it to attain the target planet. The primary difficulty in designing a gravity assist trajectory is the determination of the encounter times of the launch planet, intermediate planet and target planet. Once these times are determined, it is a relatively simple procedure to use Lambert's theorem and numerical integration to complete the detailed design. In this paper, a procedure is described for finding these encounter times using Lambert's theorem and a new criterion based on Tisserand's criterion to identify pairs of transfer orbits between the launch planet and intermediate planet and between the intermediate planet and target planet.

INTRODUCTION

Gravity assist trajectories are an important class of trajectories that have been used by Voyager, Galileo, Cassini, and other missions to tour the solar system. The design of interplanetary trajectories involves finding an orbit that will transfer a spacecraft from the vicinity of one planet to the vicinity of another planet. The accessibility of a target planet, particularly those beyond the orbit of Jupiter, depends on finding a transfer orbit with energy relative to the initial starting point, normally the Earth, within the capability of the launch vehicle. The use of gravity assist to increase the transfer orbit energy has opened up the exploration of planets that otherwise would not be accessible with current launch vehicle capability.

Most interplanetary and planetary orbiter mission trajectories since the beginning of the space age have used Keplerian two-body motion in their design. Missions requiring three-body transfers are generally limited to those involving the satellites of the major planets, for example missions to Lagrange points in the Earth-Sun system, or the Voyager mission to the outer planets. Even though gravity assist trajectories can be designed by repeated application of two-body theory, they are included in the three-body classification because the gravity assist requires a simultaneous exchange of energy among three bodies. The three-body theory employed for the design of gravity assist trajectories involves the use either of vectors defining the approach and departure hyperbolic asymptotes with respect to the gravity assist planet, or of Tisserand's criterion, which pertains to the interplanetary Keplerian orbits connecting the launch, gravity assist and target planets. It will be shown that while both design techniques follow from the Jacobi integral, they yield significantly different results, since they represent different approximations of the true equations of motion. In this paper, a criterion is developed that combines both of these design techniques.

JACOBI INTEGRAL

An important integral describing constraints on energy transfer for the restricted three body problem was discovered by Carl Gustav Jacob Jacobi in the Nineteenth century. A point mass moving in the vicinity of two massive bodies in circular orbits about their barycenter will conserve a certain function of the state and gravitational parameters of the massive bodies referred to as Jacobi's

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integral. The constant of integration is called Jacobi's constant. The equations of motion for two massive bodies and a point mass are given by,

$$\ddot{x} = GM_1 \frac{x_1 - x}{r_1^3} + GM_2 \frac{x_2 - x}{r_2^3}$$
$$\ddot{y} = GM_1 \frac{y_1 - y}{r_1^3} + GM_2 \frac{y_2 - y}{r_2^3}$$
$$\ddot{z} = GM_1 \frac{z_1 - z}{r_1^3} + GM_2 \frac{z_2 - z}{r_2^3} \quad (1)$$

The two massive bodies rotate around the barycenter and the rotation rate is simply 2π divided by the period of the orbit,

$$\omega = \sqrt{\frac{GM_1 + GM_2}{\rho^3}} \quad (2)$$

where ρ is the distance separating the two massive bodies. The geometry is illustrated on Figure 1.

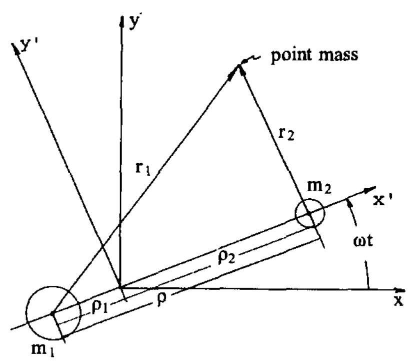

Figure 1 Restricted Two Body Geometry

The primed coordinate system (x', y', z') in Figure 1 represents a rotating coordinate system in which the two massive bodies lie on the x' axis, with

$$x = x' \cos \omega t - y' \sin \omega t$$
$$y = x' \sin \omega t + y' \cos \omega t$$
$$z = z' \quad (3)$$

After differentiating Equations 3 twice, substituting into Equations 1 and eliminating the sine and cosine terms, the following result is obtained as shown in Reference 1.

$$\ddot{x}' - 2\omega\dot{y}' - \omega^2 x' = -GM_1 \frac{x_1' - x'}{r_1^3} - GM_2 \frac{x_2' - x'}{r_2^3}$$
$$\ddot{y}' + 2\omega\dot{x}' - \omega^2 y' = -\left(\frac{GM_1}{r_1^3} + \frac{GM_2}{r_2^3}\right) y'$$
$$\ddot{z}' = -\left(\frac{GM_1}{r_1^3} + \frac{GM_2}{r_2^3}\right) z' \quad (4)$$

Equation 4 may be put into a form that can be integrated by defining the function

$$U = \frac{1}{2}\omega^2 (x'^2 + y'^2) + \frac{GM_1}{r_1} + \frac{GM_2}{r_2} \quad (5)$$

and substituting into Equations 4.

$$\dot{x}'\ddot{x}' - 2\omega\dot{x}'\dot{y}' = \dot{x}' \frac{\partial U}{\partial x'}$$
$$\dot{y}'\ddot{y}' + 2\omega\dot{x}'\dot{y}' = \dot{y}' \frac{\partial U}{\partial y'}$$
$$\dot{z}'\ddot{z}' = \dot{z}' \frac{\partial U}{\partial z'} \quad (6)$$

Adding Equations 6,

$$\dot{x}'\ddot{x}' + \dot{y}'\ddot{y}' + \dot{z}'\ddot{z}' = \dot{x}' \frac{\partial U}{\partial x'} + \dot{y}' \frac{\partial U}{\partial y'} + \dot{z}' \frac{\partial U}{\partial z'} = \frac{dU}{dt} \quad (7)$$

The integral of Equation 7 given below is called the Jacobi integral.

$$\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 = 2U - C \quad (8)$$

or

$$\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 = \omega^2 x'^2 + \omega^2 y'^2 + 2\frac{GM_1}{r_1} + 2\frac{GM_2}{r_2} - C \quad (9)$$

where C is the constant of integration.

TISSERAND'S CRITERION

Francois Felix Tisserand was a nineteenth century astronomer who discovered a unique application of Jacobi's integral to identify comets. In the restricted three body problem, a certain function of the orbit elements before and after a planetary encounter is conserved. If this function is computed for two comet observations on different orbits and the results are the same, one may conclude that the observations are of the same comet and the comet has encountered a planet between the observations. This may be confirmed by propagating the orbits forward or backward in time to see if they encountered a planet.

In the application of Tisserand's criterion to gravity assist trajectory design, the procedure is reversed. Transfer trajectories from the launch planet to the intermediate planet and from the intermediate planet to the target planet are computed using Lambert's theorem. These trajectories are matched based on Tisserand's criterion to identify viable launch and encounter opportunities. Tisserand's criterion follows directly from Jacobi's integral. Using Equations 3, the Jacobi integral is transformed back to inertial coordinates (the unprimed coordinates on Figure 1).

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\omega(x\dot{y} - y\dot{x}) + 2\frac{GM_1}{r_1} + 2\frac{GM_2}{r_2} - C \tag{10}$$

For GM_1 much greater than GM_2, the z component of the angular momentum vector is given by,

$$x\dot{y} - y\dot{x} = h_z = h\cos i \tag{11}$$

$$h = \sqrt{GM_1\ a(1-e^2)}$$

and from the *vis viva* integral the energy is given by,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = GM_1(\frac{2}{r_1} - \frac{1}{a}) \tag{12}$$

Substituting Equations 11 and 12 into Equation 10 gives

$$GM_1(\frac{2}{r_1} - \frac{1}{a}) - 2\omega\sqrt{GM_1\ a(1-e^2)}\cos i = 2\frac{GM_1}{r_1} + 2\frac{GM_2}{r_2} - C \tag{13}$$

Substituting Equation 2 for ω and for small GM_2 compared with GM_1,

$$C \approx \frac{GM_1}{a} + 2GM_1\sqrt{\frac{a(1-e^2)}{\rho^3}}\cos i \tag{14}$$

In the literature, Tisserand's criterion is often developed in dimensionless coordinates and the Jacobi constant modified to remove constant parameters. If a is divided by ρ to define $\bar{a}$ and Equation 14 is multiplied through by ρ and divided by GM_1, Tisserand's criterion in dimensionless coordinates becomes

$$\frac{C\rho}{GM_1} \approx \frac{1}{\bar{a}} + 2\sqrt{\bar{a}(1-e^2)}\cos i$$

If the first observation of a spacecraft or comet has orbit elements a_1, e_1 and i_1 and the second observation after a planetary encounter has orbit elements a_2, e_2 and i_2, then

$$\frac{1}{a_1} + 2\sqrt{\frac{a_1(1-e_1^2)}{\rho^3}}\cos i_1 \approx \frac{1}{a_2} + 2\sqrt{\frac{a_2(1-e_2^2)}{\rho^3}}\cos i_2 \tag{15}$$

GRAVITY ASSIST VECTOR DIAGRAM

Figure 2 shows the encounter geometry in the vicinity of the intermediate planet that supplies the gravity assist energy boost to the spacecraft. The incoming velocity of the spacecraft ($\mathbf{V}_1$)is subtracted from the planet velocity ($\mathbf{V}_p$) to obtain the planet relative approach velocity ($\mathbf{v}_i$) as shown in the upper vector diagram on Figure 2. The lower vector diagram shows the same relationship for the outgoing velocity vectors. If the incoming and outgoing velocities are computed far from the planet yet close enough to the planet that the heliocentric energy may be assumed constant, the velocities

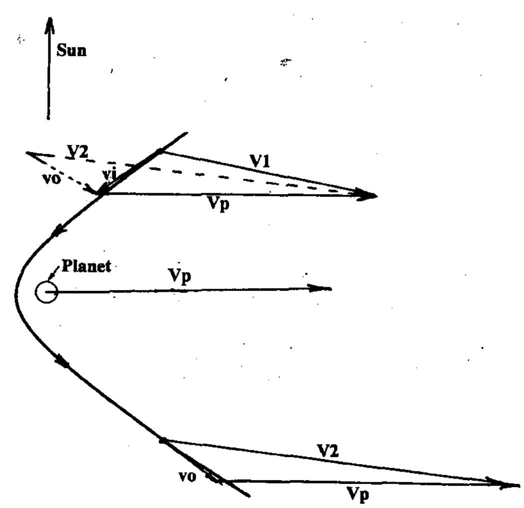

Figure 2 Gravity Assist Vector Diagram

$\mathbf{v}_i$ and $\mathbf{v}_o$ are approximately the $\mathbf{v}_\infty$ vectors associated with the two-body hyperbola about the planet. In the limit of two-body motion assumed for patched conic trajectories, $\mathbf{v}_i$ and $\mathbf{v}_o$ are equal in magnitude. Since the planet velocity is also assumed to be constant during the relatively short time interval of the planet encounter, the outgoing vector diagram may be superimposed on the incoming vector diagram as shown on Figure 2. The outgoing heliocentric spacecraft velocity magnitude is greater than the incoming velocity magnitude and the spacecraft has acquired additional orbit energy relative to the Sun. The energy acquired by the spacecraft comes from the Sun and the planet.

Consider the triangle formed by the spacecraft and planet heliocentric velocity vectors and the incoming velocity vector. From the law of cosines,

$$v_i^2 = V_p^2 + V_1^2 - 2V_pV_1\cos A \tag{16}$$

The orbit of the planet about the Sun may be approximated by a circle with velocity magnitude given by,

$$V_p = \sqrt{\frac{GM_s}{\rho}} \tag{17}$$

The heliocentric orbit of the spacecraft may be regarded as a two-body conic. The velocity magnitude is given by

$$V_1 = \sqrt{\frac{2GM_s}{\rho} - \frac{GM_s}{a_1}} \tag{18}$$

In the plane of the orbit, the angle A is simply the flight path angle (γ). For the general case, the angle A is a function of γ and the inclination of the spacecraft orbit plane with respect to the planet orbit plane i_1 and

$$\cos A = \cos\gamma\cos i_1 \tag{19}$$

$$\cos\gamma = \frac{\sqrt{GM_s a_1(1-e_1^2)}}{V_1 r_1} \tag{20}$$

Making these substitutions into Equation 15 gives,

$$v_i^2 = \frac{2GM_s}{\rho} - \frac{GM_s}{a_1} + \frac{GM_s}{\rho} - 2V_1\sqrt{\frac{GM_s}{\rho}}\sqrt{\frac{GM_s a_1(1-e_1^2)}{\rho^2 V_1^2}}\cos i_1 \tag{21}$$

The energy of the spacecraft relative to the planet, the potential energy of the spacecraft relative to the Sun and the velocity of the planet relative to the sun may be regarded as constant. Collecting these "constant" terms on the left side gives,

$$C \approx \frac{3GM_s}{\rho} - v_i^2 \approx \frac{GM_s}{a_1} + 2GM_s\sqrt{\frac{a_1(1-e_1^2)}{\rho^3}}\cos i_1 \tag{22}$$

Equation 22 provides an interesting insight into the geometrical meaning of Jacobi's constant in the limit where one of the gravitating bodies is much more massive than the other. The terms in the Jacobi integral are related to the velocity vector diagram of the participating bodies and the relationship to energy conservation is incidental.

CASSINI TRAJECTORY DESIGN

The Cassini mission to Saturn provides an example of the application of Tisserand's criterion to the design of a gravity assist trajectory. The segments of the Cassini trajectory that are of interest are from Earth to Jupiter and from Jupiter to Saturn. For the purpose of this analysis, the encounter time and initial conditions at Earth relating to energy are given.

The first step is to determine the encounter times at Jupiter and Saturn. An initial guess of the encounter times of Jupiter and Saturn is made based on the approximate flight times associated with a Hohmann transfer. Point to point conic solutions for the trajectory segments from Earth to Jupiter and from Jupiter to Saturn are computed using the solution of Lambert's theorem discovered by Lagrange. A point to point conic solution assumes zero mass for the planets and only the gravity of the sun is included. The solution of Lambert's theorem gives the two-body conic connecting two position vectors where the flight time is known. The two position vectors are obtained from the planetary ephemerides and the conic trajectory is computed from planet center to planet center as shown on Figure 3.

The next step is to compute the velocity vectors relative to Jupiter, one for the incoming trajectory segment ($\mathbf{v}_i$) and one for the outgoing trajectory ($\mathbf{v}_o$). If the Jacobi constants for the two

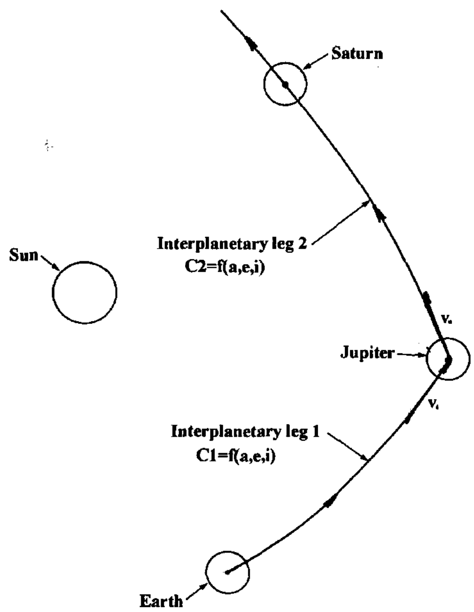

Figure 3 Earth-Jupiter-Saturn Encounter

trajectory segments do not match, then the following procedure can be used to find potentially viable encounter time solutions. The encounter time of Jupiter is fixed and the encounter times of Earth and Saturn are permitted to vary over a suitable range of times. For each pair of Earth-Jupiter encounter times and Jupiter-Saturn encounter times, a Lambert solution is computed and the Jacobi constant is computed from the orbit elements. The Jacobi constants for the two interplanetary trajectory legs are matched and the results cross plotted on Figure 4. Several approximations may be used for computing the Jacobi constant. Results for Tisserand's criterion and the Jupiter energy criterion are shown on Figure 4 as dashed lines. The Jupiter energy criterion (Equation 22) is equivalent to matching the incoming and outgoing velocity magnitudes relative to Jupiter. For this paper, a criterion is used that matches the average of Tisserand's criterion and the Jupiter energy criterion and is shown on Figure 4 as the solid line. The equation for this papers criterion, after simplification to remove constant parameters, is given by,

$$\frac{1}{a_1} + 2\sqrt{\frac{a_1(1-e_1^2)}{\rho^3}}\cos i_1 - \frac{v_i^2}{GM_s}$$

$$\approx \frac{1}{a_2} + 2\sqrt{\frac{a_2(1-e_2^2)}{\rho^3}}\cos i_2 - \frac{v_o^2}{GM_s} \qquad (23)$$

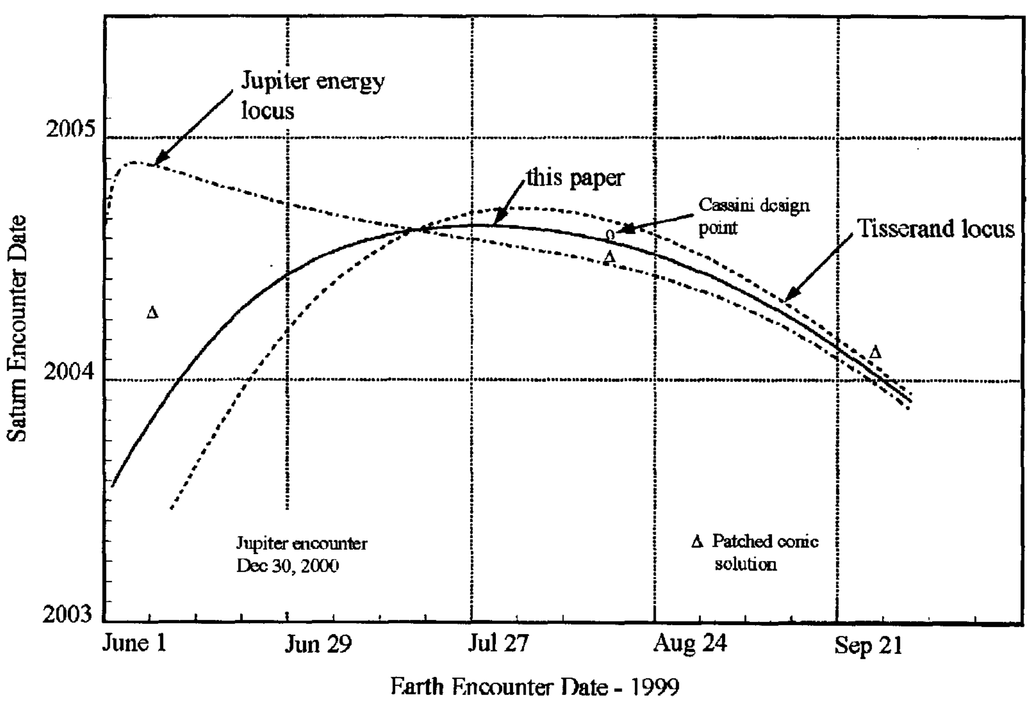

Figure 4 Earth-Jupiter-Saturn Loci

For a given pair of Earth launch and Saturn encounter times indicated on Figure 4, the approach and departure velocity vectors at Jupiter are obtained and the hyperbolic conic relative to Jupiter is computed. A preliminary assessment of the viability of the Jupiter centered hyperbola is performed. A trajectory that intersected the surface of Jupiter, for example, or hits one of Jupiter's satellites would not be viable. Next, the encounter conditions at Earth and Saturn are examined for viability. If the energy at Earth or Saturn is unacceptable, the trajectory is not viable. If a viable trajectory is not found for all the launch date encounter date pairs indicated by Figure 4, the above procedure is repeated for another Jupiter encounter time.

Once a viable set of encounter times has been determined, a patched conic trajectory is designed that connects Earth Jupiter and Saturn.

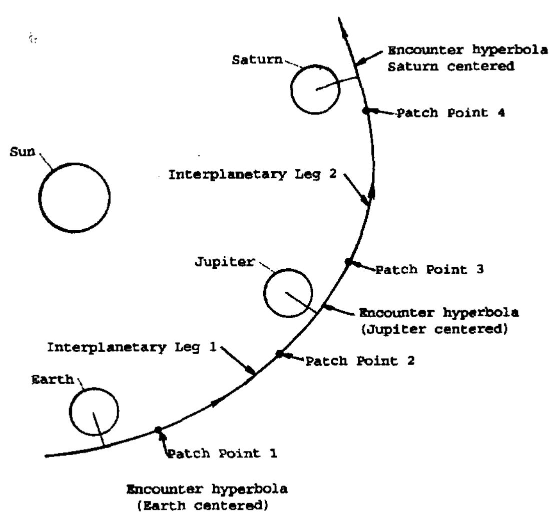

Figure 5 Earth-Jupiter-Saturn Encounter

The procedure involves computing the approach and departure velocity vectors at the patch points shown on Figure 5 from the point to point conic solution. The two-body hyperbolic trajectory is then computed with respect to each of the participating planets. For Earth and Saturn, the departure and approach target plane positions are given. A new set of patch point positions relative to the Sun are computed. The states relative to Earth, Jupiter and Saturn are added to the respective planetary ephemerides at the appropriate times. The patch point times are selected such that hte spacecraft position is near the sphere-of-influence of the planets. The planetary ephemerides may be computed from two-body orbit elements with respect to the Sun. This procedure is repeated several times for the new patch points until a ballistic trajectory is obtained from Earth to Saturn. It will be necessary to allow the Saturn encounter time to vary a small amount from the point to point solution. The results, shown on Figure 4 for three launch dates, compare favorably with the point to point solutions from this papers criterion. Also, the Cassini design point, obtained by numerical integration, is shown on Figure 4 for comparison.

The patched conic solution is used as a starting point for targeting an integrated trajectory. A comparison of the Cassini integrated trajectory and the patched conic solution is shown on Figure 6. State vectors are computed from the patched conic trajectory and differenced with state vectors obtained from the integrated Cassini ephemeris. The magnitude of the position difference is plotted as a function of time and the heliocentric range of the spacecraft is also plotted for comparison. The maximum error is less than one percent of the heliocentric range. Since the period of the Saturn orbit is 29 years, an error of several months in the predicted encounter time at Saturn from the point to point conic solutions should be expected. This error in computing the encounter times is exacerbated by accelerations from the third body that has been ignored for the two-body computations. However, a design error of only one percent enables a fairly accurate assessment of mission design constraints from the conic solution.

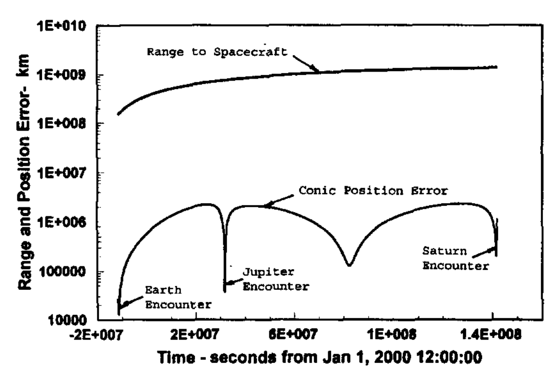

Figure 6 Cassini Conic Design Error

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